Royce's Logical Essays

Collected Logical Essays of Josiah Royce

Edited by

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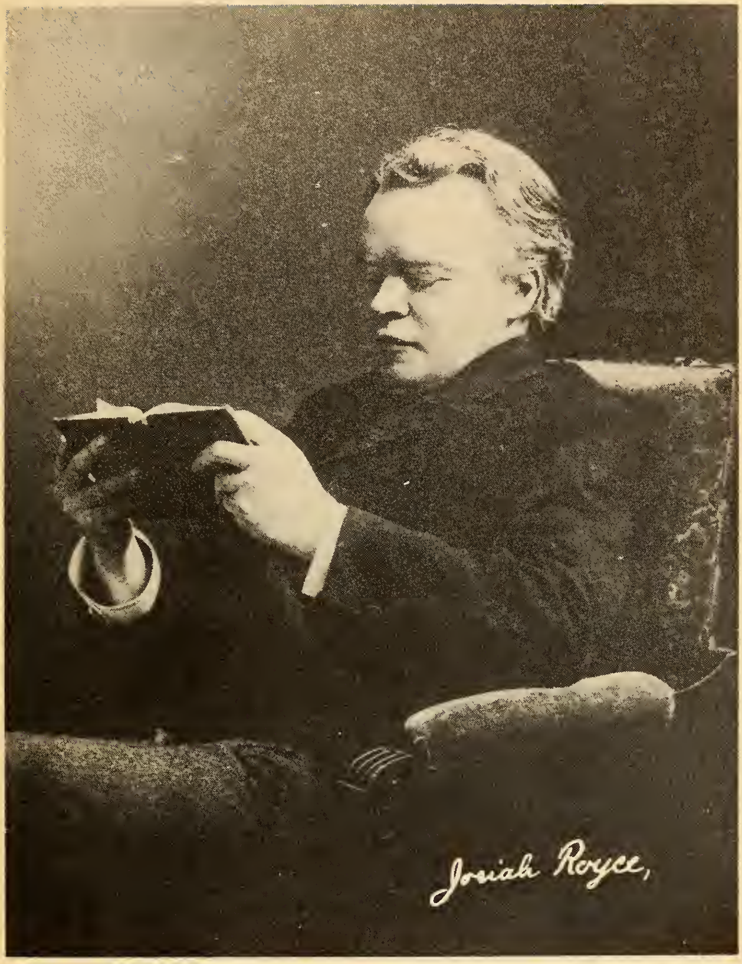

Harvard University Photo

JOSIAH ROYCE ABOUT 1900

My dear Sir

Allow me to introduce to you
Mr. Josiah Royce who has
been in the Johns Hopkins
University for two years as
a graduate student of
Philosophy & Literature and
is now called to his alma
mater as Asst. Professor
in the Univ. of California.
You will quickly discover
his interest in & acquaintance
with philosophical thought
& perhaps you will enlist
him as an occasional

writer for your excellent Journal. I commend him in a special manner to your confidence & regard.

Yours with high esteem
D. C. Gilman

Amherst, Mass.
July 18. 1878

Dr Wm T Harris

Above is a facsimile of the letter of commendation of Josiah Royce which was written by President D. C. Gilman, of Johns Hopkins University, to William Torrey Harris, Editor of the Journal of Speculative Philosophy. Royce received the Ph. D. degree from Johns Hopkins University in June, 1878, when he was twenty-two years of age. This letter is in the collection of letters by Royce to Harris, presented to the Hoose Library of the School of Philosophy, University of Southern California, by Miss Edith Davidson Harris, daughter of Dr. Harris.

Johns Hopkins University,
Baltimore, Md.
Jan. 4, 1878.

Mr. W. T. Harris,

Dear Sir;

I send by express to your address this day a transcript of an essay of my own on "The Ethical Studies of Schiller". The essay was read a short time since before the Johns Hopkins Philological Association; and at Pres. Gilman's advice I take the liberty of asking you if you can make any use of the manuscript for the "*Journal of Speculative Philosophy*." The subject is of course no new one; yet so far as I know there is no very exten-sive literature in English treating of it. In any case, Kant's influence on German Literature is a topic, it seems to me, that would bear much discussion. — If you would encourage the idea, I should like also at some future time, to offer an essay on the philosophic studies of Novalis, in which I should seek to discuss Kant's influence on the early Romanticists.

I hope I do not trespass upon your time too much in thus adressing myself to you without having had the pleasure of a previous intro-duction. After all it may not be

unwelcome to you to hear a word
of the philosophic studies that
a few at the Johns Hopkins
University are engaged in; even
though you may not find much
worth in what I herewith send.

Believe me Sir,

Very Respectfully

Yours Truly

Josiah Royce,

Fellow in Philosophy,

J. H. U.

Above is a facsimile of the letter which Josiah Royce wrote to William Torrey Harris concerning his essay entitled "The Ethical Studies of Schiller". The essay was published in the Journal of Speculative Philosophy, Vol. XII, pp. 373-392. It was written when Royce was twenty-two years old. The original letter is in the collection of letters from Royce to Harris, presented to the Hoose Library of the School of Philosophy, University of Southern California by Miss Edith Davidson Harris, daughter of Dr. Harris.

San Francisco
Oct. 23, 1878.

Dear Sir

The advance sheets of the Journal have been received. — I had a single Latin phrase in the article: "solvitur ambulando". My copyist or the printer has made it <u>sobriatur</u> ambulando. Would it be possible for you to have noted in the supplement [redacted] of the Journal ~~noting~~ this error. Otherwise the nonsense seems too glaring. But the matter is of little consequence. — I should like to purchase 30 copies of this number of the Journal, and have them forwarded by Exp. C.O.D.

Excuse haste — Yours J. Royce

Above is a facsimile of a postcard written by Royce to Harris, referring to a typographical error he had found in the essay entitled "The Ethical Studies of Schiller". The error was corrected. See Journal of Speculative Philosophy, Vol. XII, p. iv, under Errata. The original postcard is in the collection presented to the Hoose Library of the School of Philosophy, University of Southern California, by Miss Edith Davidson Harris.

EDITOR'S PREFACE

Josiah Royce was born at Grass Valley, California, November twentieth, 1855. Since the centennial year of his birth is approaching it is hoped that the publication of this collection of his logical essays may promote a revival of interest in the thinking of the ablest of all American logicians. Although he is fully entitled to this high evaluation, these logical essays have been overlooked, at least in their entirety, by many critics and expositors of his philosophy. Doubtless this is explicable by the fact that several of them were contributed to encyclopedias whose readers seldom even notice the name of the author of an article.

The editor had the good fortune to be a member of the last class in Advanced Logic which Professor Royce taught at Harvard University. Royce offered this course in the Spring Semester, 1916, and he died September fourteenth, near the end of his sixty-first year. The late Mr. Ralph Monroe Eaton was also a member of that class, and he served as Graduate Assistant in the course. His two books, Symbolism and Truth, and General Logic, rank among the important contributions to logic made by any of Professor Royce's students. However, Eaton adopted a conceptualistic interpretation of logical entities and relations, under the influence of his neo-realistic teachers who were especially critical of Royce, whereas his distinguished teacher always insisted that logic is primarily concerned with interpreting objective types of order which he thought possess a cosmic and metaphysical significance. Consequently, in both of his books on logic, Eaton refers to Royce only once, and that reference is to the self-representative system explained in the well known Supplementary Essay to The World and the Individual!

Professor C. I. Lewis generously acknowledges his indebtedness to Royce in the Preface of his important historical treatise entitled A Survey of Symbolic Logic. He writes: "Most of all, I am indebted to my friend and teacher, Josiah Royce, who first aroused my interest in this subject, and who never failed to give me encouragement and wise counsel. Much that is best in this book is due to him."

The editor's elementary logic textbook entitled The Principles of Reasoning (3rd edition, 1947. Appleton, Century, Crofts, Inc., New York) attempts to combine Royce's interpretation of order systems with Bernard Bosanquet's conception of identity in difference by using the concepts order system, implicative system, and inferential whole as synonymous designations of the ultimate unit of knowledge and reality. In that textbook emphasis is especially given to the outstanding importance of several of these logical essays.

The editor well remembers Professor Royce announcing to the class in Advanced Logic, about the middle of the Semester, that he had just received a postcard from Dr. James Hastings, Editor of the Encyclopedia of Religion and Ethics, informing him that he had six weeks to complete his article entitled Negation, and three months to complete Order. This means that these were among the very last essays that Royce wrote. Occasional comments to this class about his article published in the Transactions of the American Mathematical Society, made it clear that he considered this essay to be his most important single contribution to logic. (See Chapter XVII).

By carefully studying these logical essays of Josiah Royce students and teachers of philosophy will become intimately acquainted with a superb creative intelligence capable of stirring to the depths whatever reflective ability they may possess. For Royce was a philosophic genius whom every real thinker must at least respect, and one whom no student who aspires to become a philosopher can afford to neglect. Professor Hoernlé rightly comments: "To any young American student of philosophy who rejects Absolute Idealism I would say that he has no right to dissent or condemn, unless he has first earned that right by a thorough study and understanding of Royce." (Contemporary Idealism in America, Edited by C. Barrett, MacMillan Co., New York, 1932, p. 301).

Josiah Royce now belongs to the ages. No competent authority will question his right to be ranked as one of America's foremost logicians. Indeed, it is the opinion of the editor that this collection of his logical essays constitutes the most substantial, enduring, and original contribution to logic that has yet been made by an American philosopher.

In 1881, when he was twenty-six years of age, Royce published A Primer of Logical Analysis for the Use of Composition Students. This little book is now quite rare. So far as the editor knows it is the only important one of his logical essays that is omitted from this collection. For a bibliography of Royce's writings, the reader is referred to The Philosophical Review, Vol. XXV, No. 3, May 1916, and the addenda thereto in John Edwin Smith's Royce's Social Infinite, p. 171, (The Liberal Arts Press, 1950.) See also R. B. Perry's biographical sketch of Royce in the Dictionary of American Biography, and the editor's article entitled "Josiah Royce -- California's Gift to Philosophy" in the Personalist, Volume XXXI (1950), pp. 352-368.

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Director of the School of Philosophy
University of Southern California

TABLE OF CONTENTS

PART ONE

Non-Symbolic Logic and Methodology

PART TWO

Symbolic Logic

Part I

Non-Symbolic Logic and Methodology

Chapter I

RECENT LOGICAL INQUIRIES AND THEIR PSYCHOLOGICAL BEARINGS

Editor's Note: This article is Professor Royce's Presidential Address delivered to the members of the American Psychological Association in January, 1902, and published in Vol. 9 (1902) of the *Psychological Review.* Reprinted by permission.

The American Psychological Association has always given a kindly recognition to the general philosophical interests which many of its members represent, as well as to the more distinctively psychological concerns which properly form the center and the main body of its undertakings. In honoring me, by calling me to fill for the year the office of president, my fellow members have well known that they ran the risk of hearing a discussion rather of some philosophical problem than of a distinctively experimental topic. I, in my turn, am quite unwilling to ignore or to neglect the fact that ours is primarily a psychological association, while I am equally aware that the general student of philosophy is at a disadvantage when he tries to discuss with the productive workers in the laboratories the matters which, as their specialty grows, come to be increasingly their own peculiar possession. Yet a presidential address is properly an opportunity for studying the problems suggested by a comparison of various fields and methods of work. And accordingly, upon this occasion, I propose to discuss some questions that lie on the border land between psychology and the distinctively philosophical disciplines. These questions in part directly touch undertakings which already occupy a recognized place in the psychological laboratories. In part they seem to me to promise to yield in future still wider opportunities for experimental research than are now open. In any case they are questions of permanent interest, and of increasing importance, which neither the psychologist nor the philosopher can afford to ignore.

I.

I have named my paper a discourse upon 'recent logical inquiries and their psychological bearings.' By the term 'recent logical inquiries,' I mean to refer to two decidedly distinct classes of researches, both of which are to-day receiving much attention. To the first of these two classes belong researches directly bearing upon the psychology of the thinking process, and upon the natural history of logical phenomena in general. Such inquiries may be called logical, since they are sometimes undertaken by logicians for the sake of their own science, and in any case are suggested by the problems of logic. Meanwhile, studies of this class are obviously also, at least in intention, contributions to psychology. But I wish, in addition, before I am done, to call attention to quite another class of researches, whose psychological bearing is not at first sight so evident. This my second class of recent logical inquiries consists of studies in the comparative logic of the various sciences, and of examinations of the first principles of certain special sciences. I refer here especially to such books as Mach's well-known volume on the 'Principles of Mechanics,' and to all the large literature that has grown up about the problems suggested by the fundamental concepts of the different natural sciences. I place in the same class, moreover, the elaborate and fruitful researches into the foundations of arithmetic, of geometry, and of the Theory of Functions, which are due to such mathematicians as Cantor, Dedekind, Peano, Klein, Hilbert. The last three or four decades have seen an enormous extension of the literature of this type. I include, moreover, in the same class, certain more distinctively philosophical treatises such as Russell's 'Essay on the Foundations of Geometry,' and Couturat's volume on the 'Concept of the Infinite,' and these are but specimens of the class of inquiries in question.

I mention this vast collection of significant studies, not because I am in any sense a master in this field of the comparative logic of the sciences, but because, as a humble learner, I have been trying to make my way in some of the plainer of the paths that these recent studies have been opening, and because I hope, by a few wholly inadequate, but at least timely indications, to show upon this occasion that this relatively new comparative study of the fundamental conceptions of various sciences, is full of promise for the psychologist as well as for the logician.

Of the intrinsic importance of this my second class of 'logicial inquiries,' there can be, in many cases, no doubt. From the literature of comparative logic to which I thus refer, there is certain to grow, with time, a new science, which I may venture to call a comparative morphology of concepts. This science will occupy a borderland position. In one respect, it will belong to philosophy properly so called. For it will lead to advances in just that critical consideration of the foundations of knowledge which constitutes one principal division of philosophy. Upon the other hand, the new science will be an empirical as well as a reflective doctrine. It will include a critical examination of the history and evolution of the special sciences. And in this respect it will take its place as a contribution to the general history of culture, and will furnish material for the student of anthropology and of social psychology. And, still further, the new science will contribute to the interests of the student both of general and of experimental psychology. For it will set in a new light the empirical problems of the psychology of the intellect. It will define, in new form, issues which the descriptive psychologist must attempt to reconsider. And, as I am convinced, it will present an ample array of problems for the experimental psychologist,--problems which he alone will be able to pursue into some of their deepest recesses. This new science, then, which you and I can hardly live to see very highly organized, but which the whole century now beginning will greatly advance, will offer large ranges of what one may call neutral ground, where philosopher and psychologist, special student and general inquirer, historian and sociologist, may seek each his own, while a certain truce of God may reign there regarding those boundary feuds which these various types of students are prone to keep alive, whenever they discuss with one another the limits of their various territories, and the relative importance of their different tasks.

II.

Two distinct and very large classes of 'logical inquiries' my title is thus intended to bring at once to your attention. My reason for naming them by means of one phrase, and for considering them in one paper, is this: When you examine the first of my two classes of recent inquiries, you find that while much is now doing to advance our knowledge of the psychology of the thinking process, we have to admit that the present state of research in this field is not wholly satisfactory. The general theories about what the place of thought is in the natural history of our minds, and about the special processes of which thinking consists, are numerous; but regarded as psychological theories, they still seem for the most part loose and ill-founded. On the other hand, the special efforts to break paths into the thickets of the psychology of the thinking process by means of experimental research, have so far met with serious obstacles, have often given negative results, and in any case have been confined to the outskirts of the subject. A survey of our first class of recent inquiries will therefore suggest to us the need of looking in new directions for additional sources of aid in the study of the psychology of the higher intellectual processes. In view of this fact it may appear, before we are done, that there is a genuine promise of help towards further advances in this branch of psychology, in case we look for such help to what I have called my second class of recent inquiries in logic. These studies in the comparative logic of the sciences are at once, as I have said, philosophical and empirical studies. They are logical researches regarding the foundations of knowledge. They are also historical reports regarding the way in which our human thinking processes have worked and are working in the world of live thinkers and of socially guided investigations. To call attention, in however feeble or summary a way, to the evidence that is thus attainable regarding the natural history of the thinking process, is a purpose that may justify my necessarily very superficial comments upon this branch of my topic.

III.

And so let me next say something about the first of my two classes of recent inquiries, namely, those that are more obviously and explicitly guided by psychological motives.

The psychology of the intellect is one of the oldest branches of psychological inquiry. In Greece it began in pre-Socratic philosophy. It became prominent in Aristotle's doctrine. Both Stoics and Epicureans contributed to it. Scholasticism elaborated and modified Aristotle's theories regarding the whole province. Modern philosophy, and in particular the English psychology, began with renewed interest in the problems of this branch of mind. Thus, the psychology of knowledge was long the favored child of the philosophers, at times when the feelings and the more purely volitional aspects of mental life were comparatively neglected in their researches. In a sense this advantage of the intellectual process has continued in recent times. The psychology of association and that of perception have been steadily advancing. Attention, discrimination, and lately memory, have been experimentally studied. But on the other hand, in recent psychology, just the region where, at the outset, the interest of the philosophers was early centered, namely, the region of study of the higher intellectual processes--conception, judgment and reasoning--is the very province of psychology where progress, in any exact sense, is nowadays so slow. The difficulty of reducing the problems which, for the psychologist, arise in this region, to any form capable of exact experimental inquiry, is notoriously great, and will of course long remain so. Meanwhile, however, the actual importance which psychological methods have won in the esteem of modern writers, have led to repeated attempts to found reforms in logic upon psychological theories. Numerous are the modern works on logic wherein the psychology of the thinking process is expounded at the beginning of the whole research, or at least is made the basis upon which an author's logical doctrines depend. The great influence of Brentano's doctrine of the process of judgment upon one whole series of logical inquiries in Germany is well known, and is an example of what I mean. The earnestness with which the problem of the nature of the 'impersonal' judgments has been discussed by a large number of modern writers on logic is another example of this subordination of logical to psychological issues. For the doctrine of the 'impersonal' forms of expression is a problem of the psychology of language, and to my mind, interests the pure logician hardly at all.

Meanwhile, if psychological doctrines have thus played a large part in the books upon logic, one can hardly feel surprised to find that, in the present state of the psychology of the intellect, the theories about the higher intellectual processes which have been expounded in the logical treatises have been somewhat dishearteningly various and capricious. Concerning the processes of abstraction and conception, certain stereotyped formulas were indeed, until quite recently, pretty constantly repeated. But with the doctrine of judgment, chaos in the textbooks of logic began. Judgment was, so one sometimes said, a process of pure association of ideas, wherein the subject idea recalled to mind by contiguity the predicate idea. But no, said others, it was rather a process of Herbartian apperception, wherein the predicate idea assimilated the subject idea and forced it to fuse with itself so that they became but one idea. On the other hand it was often something much nobler; it was an active process of synthesis, not to be confounded either with mere association or with passive fusion--a constructive process wherein subject and predicate idea came to be connected by certain peculiar mental links. Yet not so; on the contrary, it was a process of analysis, whereby a given whole was divided into parts, and the subject and predicate were the products of this sundering. Or, yet again, it was no union and no sundering of ideas at all, but something quite different--an estimate about the objective value of a connection of ideas. But still once more, it was none of all these things, it was an entirely irreducible act of accepting or rejecting an idea or a complex of ideas; and upon this psychologically irreducible and primal act was founded our very conception of any distinction whatever between the objective and the subjective world. All these things judgment has been in the text-books, and this, as you well know, is not the end. And all these views have been advanced, upon occasion, as psychological theories about the process of judgment, as theories either verifiable by direct introspection or else deducible from more general doctrines about our mental processes.

In presence of such a variety of opinions, many students interested in the theory of the thinking process have tended, in more recent discussion, to choose one of two opposed directions. Either they have been disposed to relieve themselves altogether of any responsibility for settling the psychological problems, by drawing a technically sharp line between Logic and Psychology, by devoting themselves to the former, and by leaving out of the logical inquire all consideration whatever of the descriptive psychology of thinking; or else, choosing rather the psychological road, they have attempted to reduce the problems in question to some shape such as would make possible a more exact introspection of the details of the thinking process by causing these to occur under esperimental conditions. The former of these two ways of dealing with the problem of the nature of the thinking process has recently been formally adopted, amongst other writers, by Husserl, in his Logische Untersuchungen. Husserl has vigorously protested against all psychologisirende Logik. Logic, he insists, must go its own way, yet Husserl, in his still unfinished and very attractive researches, yet lingers over the problems of what he now calls the 'phenomenological analysis' of the thinking process, and his farewell, as a logician, to psychology proves to be a very long one, wherein the parting is such sweet sorrow that the logician's escape from the presence of psychology is sure to lead to further psychological complications. As a fact, I cordially accept, for myself, the view that the central problems of the logician and of the psychologist are quite distinct, and that the logician is not responsible for, or logically dependent upon a psychological theory of the thinking process. Yet I am unable to doubt that every advance upon one of these two sides of the study of the intellectual life makes possible, under the conditions to which all our human progress is naturally subject, a new advance upon the other side. I believe in not confounding the tasks of these two types of inquiry. But I do believe that a mutual understanding between the workers will be of great importance; and I feel that we need not discuss at very great length, or insist with exaggerated strenuousness upon the mere separation of provinces in a world of inquiry wherein to-day there are rather too many sunderings.

Meanwhile, as to the other way of approaching the problem of the nature of the thinking processes, namely the way of attacking them from the side of a more careful application of the methods of recent psychology, that at present, as I have said, is beset with well-known difficulties--difficulties upon which I need not dwell long in this presence. The most important thinking processes do not occur under conditions such as either the subject in the laboratory can easily reinstate at will, or the experimenter can determine for the subject while the latter is under observation. The thinking processes upon which experimenters have so far carefully worked are therefore artificially simplified ones--important, but elementary. The numerous investigations regarding the process of the perception of small differences of various types belong here, and constitute, in one aspect, a contribution to the psychology of judgment. The mental reactions upon the presentation of words and phrases, heard or seen by the subject, have been studied by Ribot and by others. Recently Marbe has undertaken to investigate experimentally the psychology of judgment, although under conditions that I have to think by no means very satisfactory. Simple computations, acts of recognition, of estimate, of naming, have also been investigated in various laboratories. But as you know, the positive and assured results of such work have been by no means all that one could wish. Especially notable has been the decidedly negative result of a good deal of this investigation of artificially simplified thinking processes. While, to be sure, the study of the perception of small differences has shown how unexpectedly complex are the psychophysical conditions upon which such judgments depend, the effort in case of even much more complex and intelligent thinking processes to find present in consciousness contents as complex as those of a rational thinker ought to be, has not met, under experimental conditions, with the success that one might have hoped for.

Ribot discovered that in many cases, when one presented to the thoughtful subject a general term whose meaning was somewhat abstract, but nevertheless familiar to him, and when one asked him what mental contents the suddenly presented term directly brought to mind, the answer was simply, 'nothing.' Marbe, dealing with trained subjects, of scientific habits of mind, made them perform and express simple acts of judgment, under experimental conditions, and asked them to observe introspectively the conscious accompaniments of these acts. He found, in general, that the subjective accompaniment of the judgment, apart from the direct consciousness of the very act whereby one gave expression to the judgment, was nothing at all characteristic, and was very often, as in Ribot's subjects, simply nothing at all. The subject in Marbe's experiments was to make a judgment of some intellectual value, but pretty easily accessible to him, regarding a certain presented content; as, for example, he was to choose which one of the two perceived objects had a given character; or he was to answer some other simple question, regarding facts or ideas presented to his attention by the experimenter. He was at once to express this judgment, by word, or by other motor process, as the case might be. He was then to report what mental accompaniments the act of judging had involved at the critical moment. The result of the experiments was to show that these well-trained thinkers responded to the situation in question in a mainly reflex fashion. They expressed their discriminations, their translations of Latin phrases, or their other simple intellectual processes, with relatively little difficulty; and all that was characteristic of the conscious process at the moment was that they observed, of course, the expressive act itself, which they chose in a conscious sense no more and no less than one chooses any other complicated reflex act of high grade such as comes to consciousness while it is carried out. For the rest, they sometimes observed fleeting states such as doubt or surprise, and various chance associated images, or suggested motor sensations, of no importance for the understanding of what it is to judge. These accompaniments of the act of judgment were merely individual accidents.

Such negative results have appeared, upon second thought, not very surprising either to Ribot or to Marbe. Ribot points out that most of the connected and significant processes of our life have to be largely unconscious, just because we are conscious only from instant to instant, while we live with reference to relatively far-off results, and while the rational connections of life have to do with long periods of time.

The organization of our intelligent conduct is necessarily, he thinks, a matter of habit, not of instantaneous insight. And a complex abstract idea, as Ribot points out, is a 'habit in the intellectual order.' "We learn to understand a concept as we learn to walk, dance, fence, or play a musical instrument. * * * General terms cover an organized latent knowledge which is the hidden capital without which we should be in a state of bankruptcy." Marbe comforts himself for his negative results with the reflection that a 'Wissen' can never be, as a content, itself 'im Bewusstsein.' The subject judging knows, as Marbe maintains, what the act means, but no conscious content directly corresponds to or embodies this knowledge. The only necessary conscious content that is present to the subject corresponds to the outward act, the speech or gesture, whereby the subject expresses his meaning, and this, in Marbe's opinion, sufficiently explains the negative result of his own experiments.

No doubt these comments of Marbe and of Ribot have a good deal of justification so far as concerns their own experiments. On the other hand, however, we cannot feel that their experiments were at all well adapted for observing the wealth of our actual thinking processes, because what they studied was not, in most cases, any process by which a thought can come to be built up in our consciousness at all. They could not thus hope to decide how far thought ever can find a peculiar or characteristic place in human consciousness. For what they both examined were relatively reflex processes that express the mere residuum of a mental skill long since acquired by their subjects. Ribot himself thought, and no doubt consciously thought, when he planned his experiments; Marbe thought, when he considered what problem to choose for presentation to his subjects. But the subject (already, in the mentioned cases, a person of relatively high training), had little or no need to think at all in a situation as simple or as familiar in its type as the one in which the experiment placed him. Therefore it was the experimenter and not the subject in whom the process that was to be studied went on. The subject already long since knew how to meet the familiar abstract term, or to translate the simple phrase, or to answer the other plain question. Either this his previous training disposed him to wait passively, upon hearing the well-known word, until he should have some reason to use it himself, or to bring it into connection with his own acts; or else just such training had prepared him (in Marbe's experiments) to accomplish the act whereby one could express a judgment upon the simple problem presented, or could otherwise easily and instantaneously show one's accustomed skill. In no such case was it necessary that any notable intellectual contents of higher grade should come to the subject's consciousness. The mechanism established by long training was ready. It responded as the training determined. Consciousness showed indeed nothing of an abstract thinking process; but then there was no live thought present to show. Ask me "What is the sum of 3 and 2?" or "Who was Washington?" and very probably I shall just then not think at all. If I am disposed, under experimental conditions, to respond to your questions, without knowing beforehand what the question is to be, I shall, upon hearing such an inquiry, respond as smoothly as if I were a wholly reflex mechanism. And very naturally I shall then have nothing to report in the way of introspective facts of a thoughtful sort. For I shall respond much as a baggage clerk at a large station calls out the names of cheques, or as a telegraph operator writer out his messages while listening to the familiar clicks of the instrument.

To say this is not to make light of experimental methods in their application to the psychology of thought, but is to show that if the problems of the psychology of the intellect are to be prepared for more effective and advanced experimental research in future, the thinking process must first, in some measure, be more fruitfully analyzed than has yet been the case, into elementary processes of a type capable of separate experimental study. On the other hand, the way in which these processes are synthesized into the richer life of concrete thinking must be discovered mainly in an indirect fashion, through an examination of the expressions of thought in the various products of the human intellect, as they appear in language, in social institutions, in the mechanical inventions and constructions which human reason has made, and in the constitution of the sciences themselves--those highest expressions of man's ingenuity. Meanwhile, as I think, a preliminary examination of these very larger expressions of the intellect themselves, may also help us to proceed further than we have yet done in the preparatory analysis of the elementary activities upon which our thought depends, and may enable us thus to open the way towards such an experimental investigation of the conscious aspects of live thinking as just now we lack.

What then is the best means to make such a preliminary analysis of the thinking process into its elements? To analyze thought by means of a study of the phenomena of language has so far been, from Plato's time onwards, the principal undertaking of those who have approached the psychological problems of the intellect from the objective side, that is, from the side of the way in which human thought has outwardly expressed itself. The logicians and the psychologists have joined in a frequent examination of the phenomena of speech. Both types of investigators have sought thus to acquire a knowledge of what the thinking process essentially is. And this sort of inquiry still prospers. A recent logician, Benno Erdmann, has undertaken elaborate studies in this field, studies that have combined the analysis of pathological facts with those experimental researches which he and Professor Dodge have made so well known. From the psychological side, and with vast resources in the way of varied materials, Wundt has also lately prepared his really wonderful volumes on language, working with all the equipment of the experimenter, the logician, and the philosopher, but carefully distinguishing the task of this recent book from that of his own earlier treatise on logic. One may say, then, that the psychology of language is indeed in a progressive state. Yet I cannot but hold that the relation of language to the thinking process has been somewhat too exclusively emphasized by many students of the subject. Thought has other modes of expression than through the forms of speech. Language has other business besides the expression of thought. Wundt's book has the merit of emphasizing the close and primary relation of language to the expression of the feelings and to the life of the will. In consequence, Wundt very decidely sets limits to the tendency either to regard the grammatical categories as essentially logical ones, or to use the psychology of language too exclusively as a means for interpreting the psychology of the thinking process. For this very reason his book rather encourages one to look elsewhere for auxiliaries in comprehending the psychology of the intellectual life.

I have thus endeavored to sketch some of the more directly psychological of the recent inquiries into the nature of the thinking process, in order to show why, despite all these various developments, I myself think that the psychologist still has much to learn from researches in other fields than those in which he has so far been most accustomed to seek for help. These other fields are the very ones which are opened by those recent inquiries in the comparative logic of the sciences of which I spoke at the outset.

IV.

Some widespread influence, it is hard to tell exactly what, has led, during the last three or four decades, to repeated, and often seemingly independent and spontaneous, efforts on the part of the students of various special sciences to undertake an examination into the first principles of their own branches of inquiry. The mathematicians say that it was the discovery of errors in certain accepted theorems or proofs of theorems which was the principal motive leading to their own modern desire for an increased rigidity of methods, and an increased clearness regarding their fundamental assumptions. A wide extension of some of their earlier conceptions, such as the conception of a function, resulted, during the nineteenth century, from the natural advances of their science. It was found that as such conceptions extended their range of application, theorems to which no exceptions had been known at earlier stages of the science became obviously of restricted application in the new fields thus opened, and often had to be restated altogether. In consequence, proofs of these theorems which had been accepted as valid in earlier stages of the science, were seen, in the light of the enlarged conceptions, to be invalid, or to be capable of rigid statement only through the addition of precise qualifications which had earlier escaped notice. Thus there arose a keenly critical consciousness about what constituted exact statement and rigid proof. Moreover, mathematicians are especially disposed by the work of their science to compare together the results of various and apparently independent sorts of inquiry. Especially is this the case when one considers the relations of geometrical and analytical science. At one time geometrical intuition, at another time analytical computation, may lead in the advancement of mathematical knowledge. The question therefore constantly arises, Which of these two sorts of inquiry is the superior in power, or in logical exactness? Such comparisons must lead to constantly renewed self-criticism passed by the science upon itself.

Again, early in the nineteenth century, the constructive imagination of certain geometers of genius initiated an examination of the foundations of Euclidean geometry which has since proved of the utmost importance as a study in the fundamental concepts of all science. Such influences long worked in a comparatively isolated way. Towards the close of the century they combined to bring about a sort of common consciousness on the part of mathematicians regarding the methods that they required of the investigator and of the expounder of mathematical truth. This common consciousness expressed itself not only in the regions where the science was advancing to conquer new territory, but in the study of the oldest, the most fundamental, simple and universally human of mathematical ideas. The concept of number is one of the earliest of human scientific acquisitions; yet it has recently been subjected to a searching logical analysis with decidedly novel and unexpected results, so that nobody can rightly judge what it is to count or to use numbers for purposes of recording measurements, unless he has taken into consideration mathematical discussions that are hardly thirty years old. The various extensions of the number-concept,--the relation between rational and irrational numbers, the relations of number to quantity, the different systems of complex numbers, the conditions logically necessary in order that number systems should be applied to the expression of space-relations,--all these topics have been reviewed from the foundation upwards; and the work still goes on. The various actual or possible conceptions of continuity, the exact meaning to be ascribed to the concepts of numerical and of quantitative infinity, the logical position of the conception of an infinitesimal, --all these matters have been reconsidered with a care and a novelty of results which no one can appreciate who has not come into closer contact with at least a few of these researches. And now what I wish especially to emphasize, is that all these analyses, while their direct purpose is logical, inevitably possess a psychological bearing. For they throw light upon the structure which the universally human processes of counting, measuring, comparing and otherwise dealing with continuous magnitudes have always possessed. They define certain of our most fundamental intellectual interests in our world of experience. They therefore not only logically clarify and in so far transform these interests, but they tend to several otherwise hidden aspects of the natural history of these interests themselves.

For instance, the logical prominence which these modern researches in the logic of arithmetic give to our general concepts of serial order, as contrasted with our more specialized quantitative concepts, involve a generalization about the nature of the thinking process that at once has a psychological application. For we learn hereby to distinguish the activities through which we have formed the conception of any ordered series of facts from the processes whereby we have learned to apply this conception in certain important, but decidedly special, cases to the task of measuring magnitudes. The two processes are different, not only logically, but psychologically. The second is a highly specialized application of the other, which is the more primitive and the simpler. The new problem that arises for the psychologist is that of the psychology of our ideas of serial order. The forms in which this problem is to be attacked with fruitful success by the psychologist must be furnished to him by the logician of mathematics. The latter discovers by analysis what concepts of order are fundamental and what ones, logically speaking, are derived; and how the more complex forms of order are related to the simpler. The solution of this logical question is of course primarily not any decision of a question of genesis. But it is the answer to the question, What forms of order, what types of serial arrangement are of the most importance in human thinking about the world of experience? This answer inevitably tells us, however, something about what is universal in the actual constitution of those habits of our organism upon which our thoughts about order depend. It is true then that to ask, What is logically fundamental in our ideas of order? is to ask not a psychological, but a logical, question. But to discover what is logically universal, as the basis of our exact ideas, is to find out a process that must be very widely represented in those organized modes of action of which our thoughts are an inner expression. Hence the result of the logician's analysis, while it cannot be directly translated into a logical theory, is inevitably the setting of a definite problem for the experimental psychologist.

As a fact, the problems of the psychology of the concept of order form a field for experimental research whose importance the whole modern logic of mathematics makes daily more obvious, while the adaptability of the problems for the labors of the experimenter is so obvious as hardly to need lengthy illustration. Psychologically speaking, the importance of the order in which facts are presented to us is illustrated by every case of an inverted letter, by every disarrangement of a familiar temporal or spatial sequence, by every instance of the illegibility even of our own handwriting when seen in a mirror. One of our earliest and principal mental interests is in the serial order of things and in the weaving of various serial orders into systems. But mathematical science is in large part an analysis of ordinal systems. Hence an advance in our analysis of the logical concept of order, and in our knowledge of its range of application, makes possible a more fruitful study of the natural history of thought than would otherwise lie within our power.

In the modern study of the logic of the space-concept, there is again a rich field where the results of the mathematical logicians suggest problems for the psychologist. I have myself been surprised to see how little interest psychologists have generally taken in the space-theories of modern mathematics. There is a remark of Klein, repeated since by a good many writers, to the effect that modern projective geometry, with its non-metrical methods, is rather a description of the properties that are most prominent in visual space, while ordinary geometry, with its quantitative or metrical concepts, is rather founded upon our experience of the space of our touch and of our bodily movements. This remark emphasizes what is indeed an obvious fact. One may pass lightly over it, and think little of it. But its significance begins to dawn when one learns something of those logical relations between non-metrical and metrical geometry which Cayley, and later Klein himself, first made prominent. Projective geometry, taken in the abstract, can be developed without the use of any conceptions whatever of metrical relations in space. In other words, projective geometry is a science of spatial order, and not at all of spatial quantity. Cayley and Klein showed how, by the use of certain (once more, very abstract and ideal) assumptions, our ordinary metrical geometry can be made to appear as a highly specialized case of this purely ordinal science. In the light of this consideration, Klein's just cited remark about the contrast between visual space and tactual motor space suggests a very interesting, although a very complex psychological problem about the psychology of the concepts of order and of quantity in their application to space. I suppose that no psychologist would admit that visual space is primarily non-metrical; and, of course, Klein did not mean that it was purely so. For the rest, visual space is obviously related to our consciousness of the results of our movements, and cannot be isolated from them, except by a deliberate abstraction. But, on the other hand, visual space certainly does present to us the facts which projective geometry isolates; while our other space experiences do not directly involve these projective facts at all. But the projective facts, as logical analysis shows, are, when taken by themselves, non-metrical, while the laws of the metrical facts regarding space are capable of being conceptually defined as very specialized cases of results, under certain ideal conditions, of the laws of a non-metrical space-world. These considerations may not prove to have important results for the psychology of our concepts of order and of measure; but as they stand, they certainly suggest genuine problems for psychological scrutiny. I wonder, then, to find them so little regarded by the psychological students of the space problems.

In a somewhat different direction various contributions to the questions about our consciousness of space have been made, within the last few years, by M. Poincaré, who has here shown, not only all the knowledge of a great mathematical investigator, but also a decided effort to translate his analysis into psychological terms. These contributions of Poincaré, following the results of Lie and others, have laid stress upon the relation between our general spatial conceptions and the mathematical theory of 'groups'; and they promise in still another way to bring to pass connections between psychological and mathematical investigations. In view of such developments, I feel that the time is approaching when no psychologist will have a right to try to contribute to a knowledge of our space-consciousness, so long as his own geometrical conceptions are still confined to those of the mathematical textbooks of his early youth. Psychological space theories must be brought into explicit relation with mathematical theories.

V.

But I must hasten from this mention of the merely mathematical investigations to a still more summary reference to similarly analytic work that has been done in other fields of the logic of science. The books of Mach, whose name I have already mentioned, are surely known to many of you. Dr. Paul Carus has proved, as editor and as director of translations, a beneficent aid to our students in this country by making literature of this type widely accessible amongst us. And you surely know the spirit of much of this modern literature of the logic of science. It is characterized, first, by a certain measure of the same sort of critical skill which has made the modern mathematicians so rigid in their methods of proof, and so critical of their first principles. To be sure, outside of pure mathematics, you seldom meet with the degree of rigidity which that science has of late so carefully cultivated; but still the spirit of watchful self-analysis, the freedom from sacred and unquestionable dogmatic presuppositions of all sorts, the willingness to consider fairly the possibility of the opposite of any once asserted proposition, are the common features which characterize Mach, Pearson, Hertz, and the other typical writers of this recent movement. Even as I have been preparing this discussion there has come into my hands the Vorlesungen über Naturphilosophie of Ostwald--a book of whose charm a reading of the first half of the lectures has already convinced me, and whose logical spirit, whatever you may think of its results, is of the most delightful and wholesome. The researches of which such literature is the representative, are characterized by a view of the nature of the thinking process which is closely allied to that which the mathematicians have gradually developed. For one thing, human thought, in the view of such modern writers, is not bound by any one definable collection of unquestionable axioms, nor yet limited in its operation by any mysteriously predetermined set of irreducible primal concepts. It is a variable and progressive process that is concerned with the adjustment of conduct to experience. In place of unquestionable axioms, one has therefore, in any science, only relative first principles, resolutions, so to speak, to treat some portion of the world of experience as describable in certain terms. The immediate purpose of any thinking process in a special science is the description of experience, and is not what used to be meant by the explanation of facts. To describe experience is to construct a conceptual model that corresponds, point for point, so far as desired, with the observed phenomena. In order to construct this conceptual model, one has to set about one's work with a definite plan of action, a plan large enough and coherent enough to cover the intended range. One's provisionally assumed first principles, or, as such writers often say, one's postulates, are therefore chosen simply, as expressions of this coherent plan of action. One constructs one's model according to these postulates, compares the results with the facts, and is judged accordingly. Meanwhile, a paucity of elementary assumptions is to be preferred, because science, as a practical activity, loves economy. Such writers use the older forms of the principle of causation either not at all, or as sparingly as they think possible,--their reason being that they are not quite sure what the principle of causation used to mean, and that they are interested only in finding such relations. But causal explanations, as formerly conceived, seem to them to have supposed the true connections of facts to be founded in something behind the scenes, which no experience could ever bring to light. Such writers therefore seem to themselves to be working in a purely positive spirit, as Auguste Comte long ago, although in a much cruder fashion, advised us work. They often, like Mach and Pearson, call themselves anti-metaphysical. Yet, as a fact, all this analysis of the structure of the thinking processes of the special sciences, and of what I have elsewhere called the world of description, seems to me to be not only in no wise inconsistent with an idealistic philosophy, but to be a most fruitful auxiliary to such an idealistic interpretation of the facts of the universe as, in another place, I have had occasion to maintain. But here is no place for considering the philosophical value of such a view of the logic of science. What I am here concerned to show is that this effort so to expound the principles of science as to make all the assumed relations between the objects of one's thought overt and exact, rather than occult and inscrutable, relations, leads of necessity to an analysis of the process of thinking which is full of psychological suggestiveness. For a similar reason, this effort to justify scientific theories solely by their success in producing conceptual constructions that correspond in definite and controllable fashion with the phenomena, leads to a sort of practical theory of the business of thinking which closely relates the point of view of the logician to that of the psychologist. For the latter must view the thinking process as one of adjustment to the environment; and he must suppose the mental motives which determine the choice of one rather than another way of thinking to be in the long run determined, as to their natural history, by the success of one method of adjustment as compared with that of another.

In consequence, I maintain that the future study of the psychology of the thinking process will have much to gain from a use of such analyses of ideas and processes as this new science of the comparative morphology of concepts will, as it further develops, bring to light.

VI.

My hastily-made catalogue of the types of researches which belong to the second of my two classes of recent logical inquires is thus, within its present very narrow limits, completed. I must still try briefly, however, to lay stress upon a very important general feature of the thinking process which all these recent researches, whether in the specially mathematical field or in the wider field of the logic of the various natural sciences, seem to have brought to clearer light. So long as logicians were largely confined in their researches to results derived from the analysis of language, the problems which they could hand over to the psychologist were principally the classic, but as I think, relatively fruitless problems, to which Ribot's and Marbe's experimental researches have been devoted--such problems as, What has one in mind on hearing an abstract word pronounced? or, What happens in my mind when I judge that A is B? We have already seen that the modern mathematical researches have prepared for the psychologist a large collection of relatively new problems relating to our consciousness of the types of serial order, and relating also to the way in which this consciousness of order in linked to our ideas of quantity, of space and of continuity in general. Many of these problems have assumed, in modern mathematical researches, decidedly instructive forms, which are now nearly if not quite ready for experimental study. But the problems which modern logical research is preparing for the psychologist are by no means limited to these. Let me call attention then to another range of problems of a very complex character, but of a type especially likely to receive, I think, ere long, a form suited to novel experimental researches.

Psychologists have already elaborately studied, in the laboratories, our consciousness of the differences between presented objects of various sorts. But a difference between two sensations, or intervals, or other presented facts, is a matter rather of perception than of more elaborate thought. We judge such a difference indeed; but the judgment occurs as a sort of more or less swift or deliberate reflex, subject to no conscious logical principles, except those implied in every least effort to attend to the facts presented, and to report accordingly. Even in such an effort, however, there appears one element that, in the life of our more familiar and complicated thinking, assumes extremely varied and important forms. The subject in a series of experiments upon just observable differences is obliged to report whether two objects appear to him to differ or not to differ in an assigned respect. Upon this side his act of judgment includes what one may call the 'yes' and 'no' consciousness, the decision as between alternatives, the selection or suppression of a certain possible response to an object. But the 'yes' and 'no' consciousness is one that is of course not limited to the case of observing small differences, but that has applications wherever we are able to judge; and one of its most important applications appears whenever we not only observe the differences of objects, but, in some more elaborate way, classify objects. Two objects, such for instance as a triangle and a circle, are in two such different classes for us (when we do judge them as figures of different classes), not merely because we observe that they are for us different in shape, but because, in the pressence of one of them we are disposed, in view of our geometrical training, and even of our purely popular habits of thought and speech, to make certain responses, to perform certain deeds, which, in the presence of the other object we should, if these deeds were suggested, suppress, reject, inhibit, as unfitting, absurd, untrue. In presence of the circle we do not only tend to follow its contour by means of certain eye movements, and to have suggested to us certain names, memories, and aesthetic impressions;

but, if we are thinking about circles we consciously accept certain of our suggested motor responses in presence of the circle, as adapted to express what it means for us, and how it is related to the rest of our life. Some of these very responses to which, in presence of a circle, we thus, so to speak, say 'yes' are amongst the ones to which, in presence of a triangle, we say 'no' in case there then arises any suggestion of our making them. Our customary summary expression of the results of many such acceptances and rejections of fitting reactions in the presence of circles and triangles takes the form of saying that 'no circle is a triangle.' This assertion is of course not the same as the assertion that our representative ideas of circle and triangle are different ideas. One's idea of a Frenchman differs from that of a dancing master. But it is absurd to say that because one is a Frenchman he cannot be a dancing master. Our assertion about circle and triangle is that they are not merely different, but belong to mutually exclusive classes. And we define for ourselves this latter fact of the mutual exclusion of the classes by means of a series of processes in which the consciousness of presented or remembered differences is bound up with the 'yes' and 'no' consciousness in a fashion that the logicians and psychologists of all ages have attempted to unravel, and that the psychologists, at least, have failed to discuss with finality, just because they have so little studied the 'yes' and 'no' consciousness, either in itself, or in its relation to our consciousness of difference.

As for the logicians, with their Eulerian diagrams, and their more recent and exact symbolic notations, they have indeed done much to clarify the more formal aspects of the conceptual relations involved in exclusions and negations; but, as Professor Ormond's paper on the place of the negative in logic showed to this association some years since, the questions here involved are amongst the most delicate and fundamental known to thought, and they are not yet closed issues. What, then, is the precise relation of the consciousness of difference to the consciousness of negation, or of mutual exclusion? Both logicians and psychologists need to study this problem more thoroughly.

But now it is just here that the modern reëxamination of the principles of the various sciences has been enlarging our ideas of the importance of the function of what I have called the 'yes' and 'no' consciousness in all our exact thinking. When I first heard about the logic of science, I was told by my teachers that the stage of a science in which it made much of classifications was a relatively imperfect stage. A science, I was told, passed to a higher stage when it learned to substitute explanations for classifications. And its explanations, in their turn, became exact whenever they passed to the highest stage of scientific knowledge, where they became quantitative. Quantity, then, was a concept of a rather mysterious dignity; but it certainly belonged to some very lofty level of thinking, where mere classifications were no longer in question. When one reached this lofty level science became mathematical, and the goal was near.

But nowadays, our new comparative logic of the sciences seems to put this whole matter in a new light. The ideal of exact special science is still mathematical, and will always remain so. But then, for one thing, mathematics, for the enlightened, is no longer merely the science of quantity, but is rather the science of exactly definable relationships of all types. Quantity itself, however, appears, in this new logic, as a conception whose properties and laws, in all the numerous branches of the science of the different kinds of quantity, are definable only in terms of the properties of certain manifolds, or complexes of ideal objects, which are called number-systems. The number-concept, which, as I before pointed out, is for the modern mathematician very prominently an ordinal concept, has become, in its various modern forms, something more general, as well as logically more fundamental, than the concept of quantity. Our exact knowledge of the laws of quantity thus tends, more and more, to appear as founded upon our knowledge of the laws of number, the latter being deeper and more universal. The result is the tendency towards what Klein has called the Arithmetisirung of mathematical methods. Now this Arithmetisirung implies in part, making prominent, as I pointed out earlier in this paper, the ordinal concepts. But it also implies giving a prominence to exactly defined classifications which I suppose has never before been known in the history of science.

Our knowledge of number-system is, in very large measure, a knowledge that there are, in each system, these and these classes of numbers, and that of every number in one of these classes one can assert what one must deny of every other number in the system Dedekind's famous and epoch-making definitions of the irrational numbers as corresponding to the totality of the classifications or Schnitte that one can make in the series of rational numbers, is one brilliant instance amongst many of the way in which classifications have become important in modern exact science. Another instance is Georg Cantor's definition of the grades, or dignities, the Machtigkeiten of infinite assemblages of objects. The discovery of this new concept by Cantor seems to me one of the most brilliant feats of constructive imagination in recent times. It has enriched mathematics, and will enrich future philosophy, with wholly new views of the problem of the infinite. Yet it turned upon a beautifully simple application of an exact principle of classification. Modern algebra, in the conception of what are called 'domains of rationality,' has again used an obvious and fundamental principle of classification, whose application to systems of numbers is very vast, and whose value in very various sorts of problems appears to be immeasurable. The most modern researches into the principles of geometry, and of the other exact sciences, in their efforts to find a sufficient and closed system of mutually independent first principles, have shown how much is gained by exactly classifying the ranges, or domains, to which various principles can be said to apply. Even the single principles, taken by themselves, appear, when thus examined, to be simply classifications of facts. Thus the principle that any two points in a space determine one straight line, while two straight lines can have but one point in common, is for certain purposes best stated as a classification of the points of space. The points namely are such that, if you choose at random any two of them, these two determine one class of points such that every point in space either belongs or does not belong to that class, while no two classes so determined have more than one point in common. Thus stated, the principle regarding straight lines and points appears as it ought to appear; namely, it appears as no self-evident axiom, but as a surprising and even baffling property of the points in space, and so as an arbitrary fact of our spatial experience. It is as if you said: "There is a nation of men somewhere such that any two men in that nation belong to one exclusive club, to which every other man either does or does not belong, while no two such clubs have more than a single member in common." Such a nation would have a strange sort of club-life. But just as such an assemblage are the points in space.

Classification from such a point of view reigns then everywhere on the highest level of exact science. Sharp classification is the goal as well as the beginning of the thought that gets embodied in the special sciences. To say 'yes' or 'no' to the question: "Does this object belong or does it not belong, for this purpose, to this collection of objects?" is the last as well as the first task of the human thinker in all his dealing with particular facts. Now the logical interest of this generalization about the nature of science lies in the consideration that, from this modern point of view, for which the special sciences, as you remember, are descriptions of phenomena, all our valid explanation of facts, just so far as they are valid, all our knowledge of the laws of nature, all our quantitative insight into things must be reduced merely to such classifications of facts, and to serially ordered systems of such classifications. Of such materials our conceptions of what I have called our world of description must consist. One modern writer has explicitly made this very generalization. I refer to Mr. A. B. Kempe, in his paper on the 'Theory of Mathematical Form.' Mathematics, according to Mr. Kempe, who illustrates his notion in a very varied way, is purely a science of exact classification, and is nothing else. It defines the relations of objects and systems of objects by classifying certain of these objects, or certain pairs, triads, or other groups of these objects, by placing certain of them together, and by distinguishing them from other objects or assemblages of objects. Thus, according to Mr. Kempe, one studies geometry in a strict logical order by beginning with the conception that the points of space are, as mere points, undistinguished one from another. One then goes further and notes that not only all points, but all pairs of points in space, may be regarded as undistinguished from one another, so long as you ignore the notions of direction and distance. One next observes, however, that if one takes account of triads of points, one has forthwith a classification of such triads, because all collinear triads of points are distinguished from all non-collinear triads. Upon the basis of this primal classification, as Kempe holds, all the rest of geometrical knowledge can be built up by adding further classifications as new principles are introduced. Every new principle means merely a new classification. And this procedure, as Kempe holds, is typical of the processes of exact thought everywhere. Science, then consists altogether of classifications.

Now what I want to point out is the enormous importance that such considerations give to the function which, in the life of our thinking, I have called the 'yes' and 'no' consciousness. This, I have said, is the consciousness wherein we are aware of accepting or inhibiting certain acts--acts through which we treat two or more objects as belonging to one class, or as belonging to classes that exclude each other. The contrast of X and not-X is always a product of the working of such a 'yes' and 'no' consciousness. Now I have said that psychologists have too much neglected the closer study of the 'yes' and 'no' aspect of consciousness. Psychologically speaking, it is that aspect of our mental life which accompanies our attitudes of readiness to perform certain deeds, and of attendant readiness to inhibit other deeds. Here then is a place where the modern logical inquiries counsel the psychologist to undertake a more careful study.

As a fact, classifications depend, for us, upon inhibitions, and upon becoming conscious of our inhibitions, and also upon bringing to notice the positive motor tendencies that are in us correlative to these inhibitions. Those who have studied abstract ideas as Ribot has done, or judgments as Marbe has done, have therefore attacked the problems of the thinking process at the wrong end. They have tried to examine the corpse of a dead thinking process. They have found little left but a reflex act. Live thinking is the process of classifying our objects by suppressing, in their presence, certain of our possible motor acts, by welcoming, emphasizing, or letting go certain of our other acts, by becoming aware, somehow, i.e., in some conscious terms, of these our positive tendencies and inhibitions, and by them regarding the objects in the light of the deeds that thus we welcome or suppress.

The most promising problem about the whole thinking process which is thus suggested to the psychologist may then be defined as this: "In what way, to what extent, and under what conditions, do we become conscious of our inhibitions?" Plainly the negative principle in consciousness, the Geist der stets verneint, is the constant accompaniment of all our higher, our organized, our thoughtful activities. It is the principle which makes exact classifications possible. And descriptive thought, in the light of these modern researches, means exact classification, and means nothing else so much. It is by contrast with our inhibitions that our positive motor processes get their precise conscious definition, as inhibitions of inhibitions, as tendencies to act by means of overcoming opposing considerations, and as assertions that are at once coordinate with, and opposed to, denials. Our abstract ideas are products of such an organized union of negative and positive tendencies. We can therefore understand the psychology of live thinking processes only in case we understand when, how far, and under what conditions, inhibition becomes a conscious process.

But now the psychology of the inhibitory processes--how vast a range of interesting phenomena, and how imperfectly explored a territory, does not this name suggest to us all? The world of the phenomena of primitive tabu, how fascinating it seems! Yet with tabu human thought about certain of the exact classifications, both of conduct and of truth, would seem to have begun. The pathology of our inhibitory consciousness, how interesting its complications--how important clinically--how significant from the humane point of view! Some years since, in a paper on the case of John Bunyan, I tried to present to the members of this Association an instance of the descriptive psychology of an experience largely made up of pathological inhibitions, occurring in the early manhood of a great genius. You all know how rich is the clinical material for the study of such cases. But the experimental psychology of the consciousness of inhibition--here surely is another extensive, accessible, and comparatively much neglected, and at the same time perfectly definite and promising field of work. I have now tried to show you that modern logical inquiries, in emphasizing the central significance that the process of classification possesses in all grades of our thought, have made more evident than ever that upon an understanding of the psychology of inhibition must depend a great deal of our further advance in a knowledge of the psychology of the thinking process.

I conclude then by urging upon my fellow members (1) the problem of our inhibitory consciousness and (2) the before-mentioned problem of the psychology of our ordinal concepts, that is, of our consciousness of ordered series of objects, as the two great tasks that are set before the students of the psychology of the thinking process by the results of modern logical inquiry.

If anything that I have said shall tend to further the mutual understanding between workers in psychological and in logical research, I shall be amply repaid for my efforts in trying thus to state to you something of what I see in the present situation of logical inquiry; while you, I hope, may in that case be not wholly unrepaid for the tediously abstract and lengthy road over which, by your kindness, I have been privileged to lead you.

Chapter II

THE MECHANICAL, THE HISTORICAL, AND THE STATISTICAL

Editor's Note: This article is reprinted by permission from *Science*, N. S. Vol. XXIX, April 17, 1914, pp. 551-566. In the opening paragraph Professor Royce explains the circumstances under which the original address was delivered at a meeting of Harvard University professors.

I. PRACTICAL PURPOSES OF THIS MEETING

This meeting is the outcome of conversations which resulted from the recent book of Dr. Henderson on "The Fitness of the Environment." Yet this company is not called for the sake of discussing, on the present occasion, that book, or any of the scientific problems which it more directly considers. The connection, then, between Dr. Henderson's book and this evening's undertaking needs some explanation. As you know from the wording of the call to which you have so kindly responded, one principal purpose which I have in mind as I address you is practical. I shall ask you, before the evening is done, to give some thought to the question: Is it advisable for us to meet again occasionally, as opportunity offers, in order to discuss some questions of common scientific interest? You represent various departments of research. Is it worth while for you, or some of you, at your own pleasure, to come together in such a way as the present one, in order to take counsel about different problems which belong, not only to a single science, and not only to some special group of sciences, but also to the realm which is common to a decidedly wide and varied range of scientific inquiries?

My part in this evening's discussion is determined by this practical question. I can not come here as a representative of any one department of research in natural science. I am limited in my present undertaking to such an appeal as a student of philosophy may have a right to address to a company of scientific men, when he wishes to ask them a practical question whose answer concerns them all.

The only justification which I have for addressing you is that the habits of a student of philosophy, and, in particular, of a student of logic, makes him sensitive to the value of a comparative scrutiny of the methods, the conceptions and the problems of various sciences.

If the main topic of the evening is a question relating to the practical value of some new mode of cooperation, in which a number of representatives of different departments of scientific research are to be asked to take part, the student of philosophy may possibly serve as a sort of travelling agent. For the kind of cooperation to which I have been asked to invite your attention would involve, if it succeeded, certain journeys which some of you might thereby be induced to make into the provinces of your colleagues. Widely traveled though all of you are, these journeys may lead occasionally to novel incidents, and may please or arouse you in new ways. My business, I say, is to act this evening merely as such a tourist agent, describing and praising as I can the new kind and combination of journeys to which my agency proposes to invite you.

Philosophy itself, in so far as it is a legitimate calling at all, may in fact be compared to a sort of Cook's bureau. Its servants are taught to speak various languages--all of them ill--and to know little of the inner life of the numerous foreign lands to which they guide the feet, or check the luggage of their fellow-men.

But if new comparative studies of the ideas of various and widely sundered provinces of research are to be carried out at all, Cook's agents, tedious as they often are, have their part to play. Regard me, then, if you wish to vary the name, as representing this evening some bureau of university travel.

II. PRELIMINARY VIEW OF THE THEORETICAL PROBLEM OF THIS PAPER

Speaking seriously, let me say that my task, upon its theoretical side, involves undertaking to present to you, in a perspective which may prove to be not wholly familiar, an outline sketch of certain conceptions and methods which actually belong to widely various sciences. These conceptions and methods in some measure concern you all, and, in our day, they are undergoing various changes, and are being applied to new problems.

The problems of each science are its own affair; but they also concern the whole body of scientific workers. To look over a somewhat wide range of scientific work, not for the sake of contributing to the researches of any one special science or group of special sciences, but for the sake of studying for their own sake some of the most general ideas and methods that are used by various scientific workers--this is, at the present time, a legitimate undertaking, and, in view of what has already been done, and is now under way, is not a hopeless undertaking.

The perspective in which such a study may place the problems of other people may help them to understand one another better. My task on its theoretical side is limited this evening to a few such general methodological remarks. These remarks may then lead us back to our practical question.

III. THE PROBLEM OF VITALISM

The name vitalism is often given to those doctrines which have used the hypothesis that the phenomena of living organisms are due to some process which is essentially identical in its nature with the process exemplified by our own conscious voluntary activities. We deliberate, plan and choose. It seems to us as if certain things and occurrences in the world are due to these our plans and choices, and are different from what they would be were our will not a factor in the world-process. On the other hand, some things and events in the natural world--notably the recurrent movements of the heavenly bodies, and the processes which attend the workings of machines, seem to us to be, in some or in many respects essentially different from the processes which result from our plans, our choices and our voluntary deeds. What is called a mechanical theory of nature, or, more generally still, materialism, undertakes to account for the vital processes, for the activities of organisms, by supposing that they too are not essentially different from the other material processes, and that they really exemplify the same natural laws which the movements of the heavenly bodies and the workings of machines illustrate.

The contrast between vitalism and materialism is, in the history of science and of philosophy, very ancient. The Greeks began with doctrines which were, in a somewhat confused way, both materialistic and vitalistic. The natural world was viewed as, in one of its aspects, a sort of machine, a chariot whose mechanical movement was an essential feature of its very being. The natural world was also regarded as through and through alive--a world of love and strife, of mixing and of sundering, of wisdom and of something resembling contrivance.

To this early Greek vitalism, which had various forms, the materialism of Democritus opposed a mechanical theory of nature which was much more ingenious and considerate than were the earliest forms in which the machine-like aspect of nature was described. On the whole, however, vitalism, the doctrine that nature acts not in vain, but in an essentially planful and designing way, was predominant in Greek thought.

The greatest Greek vitalist was Aristotle. Materialism remained in the background of ancient thought, and was destined to be revived, and to take on the form of the modern mechanical theory of nature, only after the beginnings of the new science in the seventeenth century of our era.

These ancient problems as to whether nature is rather a mechanism or an expression of something which essentially involves or resembles wisdom and contrivance, are certainly not questions which belong to any one natural science or group of natural sciences. From time to time, however, they come nearer to the surface of popular or of scientific discussion. The present is a moment when a certain interest in various forms of vitalism has once more become prominent in the discussions not only of philosophers and of leaders in popular inquiry, but of some professional students of the natural sciences of life as well.

I do not know how far it will prove to be interesting or profitable for you, as scientific men, to discuss, in your future meetings, if you have any future meetings, problems directly connected with vitalism, or with its old opponent, the mechanistic theory of the nature of life. I know only that when we mention such problems we call attention to one of the ancient boundary lines, or, as one may say, to one of the beaches where, in the realms of inquiry, sea and land come face to face with each other; so that two widely contrasting realms of nature here seem to clash. Here, then, the waves of experience tumble, and the tides of opinion rise and fall. Here, then, for that very reason, and especially at this very time, new discoveries are likely to be made in especially impressive ways.

If you are to compare notes, it will therefore not be surprising to find that questions about the relations, the contrasts and the connections of life and of mechanism will become prominent in your discussions. My own preliminary remarks on the classification of scientific methods may well be guided, then, by some interest in the scientific processes which go on upon this old boundary line--this sea-beach--of opinion and of investigation, where the vast and doubtful seas of inquiry into the phenomena of life encounter, as it were, the firm land where the mechanical view of nature finds its best known illustrations.

IV. THE VITALISM OF ARISTOTLE

It will help us in our survey of our problems about the contrasting ideas and methods followed by the inorganic sciences on the one hand and the sciences of life on the other hand, if we next say a word about one aspect of Greek vitalism which is frequently neglected.

Life-processes in general resemble our own voluntary human processes, as we have said, in so far as any living oranism seems to us as if it were guided by some sort of design, and as if, through a kind of wisdom or contrivance it adjusted means to ends. To say this, however, and even to believe that this seeming is well founded, and that, in some wise living nature really is planful, and does embody something of the nature of will, or of purpose--to assert all this is not yet to decide how close the real resemblance is between the teleology of nature and the choices and contrivances of a man who is planning and who is exerting his will.

As a fact there have been many vitalists who thought nature, and in particular organic nature, to be purposive, but who did not believe that nature is clearly aware of her own designs.

There have been many vitalists who conceived of nature as in some sense even divine in its skill, but who did not accept theism either in its primitive or in its more cultivated forms. The design argument in its later theological formulations is not any classic argument for vitalism. All this becomes manifest if you look for a moment at Greek vitalism, and, in particular, at the vitalism of Aristotle.

The Greek vitalists well knew that nature, however wise she seems to be, does not show signs of deliberating like an architect before he builds a house, or of piecing together her works as a carpenter devises a chest or a bed. For the Greek vitalist, and, in particular, for Aristotle, nature fashions, but not as a human mechanic fashions--piecemeal and by trial and error.

Nature's skill is (so such vitalists think) more like that of a creative artist, who does not pause to know how he creates. If ideas inspire the artist, he does not reflect upon what they are. Just so, while the being whom Aristotle calls God, who is conceived to exist quite apart from the world, is indeed self-knowing and is wisely self-observant, Aristotle's God is not the God of the later design argument. For he neither creates nor fashions the natural world. Nature, in Aristotle's opinion, is not God and is not God's handiwork, but is, with a certain instinctive and unconscious wisdom, a sort of artistic imitator of God's wisdom. And this natural process of imitating the divine perfection by quickening a material world with a tendency to be fashioned after a divine pattern--this process constitutes the life of the natural world.

The designs which nature expresses are therefore for Greek vitalism not the conscious designs of anybody--either God or man. They are the creative tendencies which embody themselves in the material world, by a process which we can best compare with the workings of instinct or of genius.

Now modern vitalism is far away from its Greek forerunners, but whenever, for any reason, vitalism becomes afresh interesting to any group either of philosophers or of scientific workers, it is well to remember that the contrast and the conflict between a mechanical view of nature and vitalistic view has hardly ever been limited to the decidedly special and artificial antithesis between blind mechanism, on the one hand, and conscious or deliberative design, on the other hand. For even our human art is, as Aristotle remarks, partly guided by a skill which is not conscious and is not deliberate. That which, in recent years, Bergson has called élan vital--the creative vital power, was well known, in their own way, to the Greeks.

Different as Bergson's vitalism is from that of Aristotle, the ancient view and Bergson's vitalism have in common the belief that life means a process of which the instinctive skill and the artistic genius of man give examples. The problem of vitalism is always the problem as to how such unconscious skill, such undeliberative art is made possible.

And so, even in this sketch of the varieties of scientific method, I shall in passing name to you one way in which some of the newest hypotheses may enable us to face, and perhaps in some measure to clarify, the problem as to how this stimulation of conscious designs by processes which are themselves unconsciously or, so to speak, blindly wise, is a possibility in the natural world.

V. THREE TYPES OF KNOWLEDGE: THE HISTORICAL, THE MECHANICAL AND THE STATISTICAL

So much must suffice as an introductory word regarding those problems about vitalism and mechanism which have recently been revived, and have brought us together. Herewith we are ready to proceed to our classification of the conceptions and the methods which may be used in dealing with such a range of problems as is this.

The attempt to sketch in a preliminary way what these conceptions and methods are can be preserved, I think, from vagueness, if I begin by using the guidance of a man of whom you all are accustomed to think as a true natural philosopher--one who was possessed of a very exact sort of scientific knowledge, and who was a great scientific discoverer. He was also very fond of a comparative study whereby he lighted up his own researches through thoughts that came to him from far-off fields. I refer to Clerk Maxwell. In a paper whereof some fragments are printed in his biography, as well as in various remarks in his published writings, Clerk Maxwell more than once used the classification of scientific knowledge which I shall here employ for our present purpose. Natural science, in so far as it studies the processes of the natural world, has three kinds of objects with which it deals. And it adjusts itself to these three kinds of objects by methods which, in each of three fields thus defined, vary widely from one another; while in each of the three fields both the conceptions and the methods used have much in common, and much too whereby each of the three fields differs from the others. The three sorts of objects are: (1) Historical objects, (2) mechanisms, and (3) statistically defined assemblages. The three sorts of methods are: The historical, the mechanical and the statistical.

Clerk Maxwell's few but momentous observations upon these three fields of scientific knowledge have a beautiful brevity, and show a fairly poetical skill of imagination whereby he finds and expresses his illustrations both of scientific ideas and of methods. I can not follow the master in his own skill. And I shall be unable to use his language. I must portray his classification in my own way, and must use my own illustrations.

If you wish to come into closer touch with this aspect of the master's thought, you may use the concluding passage of his famous elementary treatise on the "Theory of Heat," and several remarks in his article on "Atoms" in the ninth edition of the Encyclopedia Britannica. In addition I may refer you to the citations made by Theodore Merz in the eleventh chapter of Volume II. of his "History of European Thought in the Nineteenth Century" (pp. 599, 601 and 603).

Let me briefly review, with a few illustrations, this classification of the three fields and the three methods of natural science.

Science deals either with substantial things (such as atoms or organisms) or else with events. Let us confine ourselves here to the works of science in its dealings with natural events and processes. Science deals with the historical when its objects are individual events or complexes of events, such as is a single solar eclipse, or such as is the birth or the death of this man, or the performance of just this act of choice by this individual voluntary agent.

Science deals with the mechanical when its objects are the invariant laws to which all the individual events of some field of inquiry are subject, and when such invariant laws actually exist, and can be used to compute and to predict actual events. Thus, if the acceleration which every individual body belonging to a system of material bodies undergoes depends at every instant, in an invariant way, upon the spatial configuration of the system of bodies at just that moment, the system is a mechanical system--such, for instance, as a system of bodies moving in accordance with the Newtonian law of gravitation.

Science deals, in the third place, with the statistical, when it studies the averages in terms of which aggregates or collections of events can be characterized, and when it considers not the invariant laws, but the always variable possibilities that these averages will be subject to certain uniformities, and will undergo definable changes.

In brief, the object of historical knowledge is the single event, occurring, in the ideally simple case, to an individual thing. A free-will act or an observed eclipse serves as an example. The object of mechanical knowledge is the unchanging natural law under which every event of some type can be subsumed. Sometimes the object of mechanical science may be an individual event, but only in so far as, like the eclipse, it can be predicted by means of such an invariant law. The object of statistical knowledge is not the single event and is not the invariant law but is the relatively uniform behavior of some average constitution, belonging to an aggregate of things and events, and the probability that this average behavior will remain, within limits, approximately, although always imperfectly uniform.

VI. APPLICATIONS OF THIS CLASSIFICATION

In view of this classification of the objects of scientific knowledge, you may see at once that the issues between such doctrines as vitalism and a mechanistic account of nature appear, from the point of view of Maxwell's classification, in a somewhat unfamiliar perspective. For one need no longer merely contrast two views, the mechanical and the vitalistic. One now has a third and a mediating point of view to compare with both of them. The result is instructive.

Vitalism, whatever else it involves, always makes prominent some aspect of nature, and in particular some aspect of organic nature, such that this aspect is supposed to be, in some individual case, strictly historical. If an organism is due to a purposive process, if the reactions of an organism are, in any instance, events of the nature of conscious or of subconscious deeds--then something unique, historical and novel occurs whenever one of these vital processes is exemplified by an individual event.

On the other hand, if the mechanistic view of nature can exhaustively express the real facts, then the only natural events are of the type which the eclipses exemplify. The single events are, so to speak, points on a curve, selections from an ideal continuum whose constitution is definable in terms of an invariant differential equation.

But the third or statistical mode of viewing nature takes account of another aspect of the processes of nature. The world of the statistical view still contains events supposed to be unique and individual; but from the statistical point of view the main interest lies no longer in each event as it occurs, nor yet in its unique character. The statistical interest is directly concerned with a set or aggregate of events, with a discrete multitude of occurrences. These occurrences may prove to be examples of law. The statistical view is deeply interested in finding that they are examples of law. But the law for which the statistical method seeks is no longer a law that is ideally stable in terms of an invariant differential equation or in terms of any other timeless invariant. When found, the statistical law is an account of a collection of facts in terms of averages involving many events.

This account takes some such form as saying: "The average magnitude or velocity or size or range of the events of the class C is approximately V." Or, again, the statistical view succeeds when we can say: "A proportion which is approximately p of the events of the class a have the character b." Or finally, one expresses the statistical view when one is able to assert: "There is a probability q that c differs from d by not more than such and such an amount,--say X." All such generalizations, where the objects in question are living organisms, relate to events, but neither to merely historical single events nor to events subject to fixed laws. The statistical laws are probable and approximate laws about numbers of events.

Laws and probabilities, stated in some such form as the one just suggested, constitute the characteristic formulas of the statistical view of nature.

It is easy to illustrate how the statistical view contrasts with both the mechanical and the historical point of view by considering how each point of view applies to an event such as is expressed by the assertion: "A killed B."

For a strictly historical point of view this event, this homicide, is an unique occurrence--possibly a free-will act. It falls under moral and criminal laws, but these relate only to its value and its legal consequences. The interest of the case for a judge or a jurly lies in its novelty--and in its uniqueness. For a strictly mechanical view of things the killing resembles an eclipse. Unique as it is, it is supposed to have been essentially predictable. Perhaps if you had known the precise configuration and the accelerations of all the physical particles in the world at some appropriate moment, then this killing could have been calculated in advance. It is a mere case of a law--an eclipse, so to speak, of some sun--a point on some curve.

But for a statistical view the single killing of B by A is an event against which an insurance provision could have been made in advance--not because any mortal could have predicted whether or no A would kill B, but because the death-rate of men of B's age and occupation can be statistically known with an approximate and probable accuracy, so as to make a policy insuring B's life a contract whose value is calculable, not on mechanical but upon statistical grounds.

Now you will easily recognize that the actual knowledge of vital phenomena which science possesses is, in the main, a statistical knowledge. It is the sort of knowledge which the mortality tables of the insurance companies exemplify. We know little of the history of individual organisms, and less of their mechanism, but we can and do study the statistics of groups of organisms. In such statistical terms heredity and variation are now constantly investigated. In such terms growth and disease, as well as death, economic prosperity and social transformations, financial and political processes, the geographical distribution of organisms and the gradual accumulation and change of the material as well as the mental products of civilization--in such statistical terms, I say, all such things come to be the objects of scientific description and explanation. To give an account of the special phenomena of life in terms of mechanism remains in practise a remote ideal, despite all the proofs that the vital processes, being subject to physical and chemical laws, must be, in some sense, if not wholly, then very largely mechanical in their nature. Life may be a case of mechanism; but its phenomena are best known to science in terms of statistical averages, of laws which hold approximately true regarding these averages, and of probabilities which are definable in such terms as are used when the insurance value of a life-policy is computed. The logic of the insurance actuary is essentially the same as the logic which is consciously or unconsciously used in dealing with all forms and grades of vital processes.

This general rule regarding the methods of the sciences of life is well known to you. For it is also known that, just as a mechanical theory of the details of the phenomena of life still remains a remote ideal, so too an historical knowledge of the individual events of the life of an organism is something which may possess upon occasion great moral or social or perhaps clinical interest, but can occupy but a part, and usually a very small part, of the interest of the sciences of life.

Into the study of human history itself, devoted as such a study naturally is to the sequences of individual events, natural science enters in so far as something of the nature of statistical knowledge is acquired. And therefore the use of deliberately statistical methods in historical study, the use which Dr. Woods has recently proposed--such a use, I say, is in principle nothing essentially opposed to methods long since inexactly and unconsciously employed. For the historiometry of Dr. Woods is in principle a legitimate extension and a logically legitimate refinement of the long since well-known disposition to explain human history in terms of "historical tendencies" and of "historical forces."

In fact, the term tendency is, in every exact usage which you can give it, an essentially statistical term. To say that a has a tendency to lead to b is to declare that a more or less certainly and definitely known proportion of events of the class a are followed by events of the class b.

To introduce statistics into historical study is simply to try to make some such assertions about tendencies exact.

The constant extension of the use of statistical methods in all the sciences of life is something as familiar as it is momentous. Its very familiarity, in fact, tends to blind the minds of many to its real importance. In truth, the statistical view of nature has a logic of its own. Its three fundamental conceptions, that of an average, that of approximation and that of probability, are indeed not the only fundamental categories of our thought, but they are conceptions which go down to the very roots of our own intelligence as well as of our voluntary activity. It seems increasingly plausible to assert that these three conceptions, while they certainly have their special province, still, within that province go down to the roots of that nature of things which our sciences are studying. At all events, I find it hard to exaggerate the importance of those methods and of these ideas of natural science which are definable in terms of approximation and of probability, in the modern sense of those terms.

When Clerk Maxwell made his threefold classification of scientific methods, he did so with his eyes well open to the fact that by the statistical view of nature, and by statistical methods in science he meant something much wider and deeper than is the mere commonplace that statistical tables can be made by the census bureaus, and can be used by the insurance companies, or applied to the discovering of various special laws of nature. Let me remind you of what Maxwell had in mind.

VII. THE STATISTICAL VIEW IN PHYSICS

Clerk Maxwell was a physicist. His greatest treatise was that upon electricity and magnetism. The theory of electricity and magnetism follows methods which illustrate the mechanistic way of dealing with the problems of nature. Maxwell defined a system of differential equations in terms of which certain elementary electro-magnetic processes can be expressed. Assuming these equations to be true, one can compute the consequences of one's hypotheses, as Newton computed the consequences of supposing the law of the inverse squares to be true for a field of gravitative force. One can then compare the computed results with experience, and upon such computation and comparison with experiment one's method in this case depends. Such is an example of the essentially mechanical view of nature.

But Clerk Maxwell, working as he did at a time when the general theory of energy was in its period of most rapid development, was not content to confine himself to problems of the type of the theory of electricity. He also had his attention especially directed to those physical processes which are illustrated by the diffusion of gases, by the irreversible tendency of energy to pass over from available to unavailable forms, and by various analogous phenomena which can not be expressed in terms of the classic types of mechanical theories.

Following the initiative of Clausius, but developing along lines of his own, Maxwell thereupon worked out his kinetic theory of gases. It is that theory of which he is thinking when he distinguishes the statistical way of viewing nature both from the historical and from the mechanical view.

In fact, when the kinetic theory of gases first defines its swarms of molecules, with their countless paths and collisions, it appears to be viewing a gas simply as a complex mechanism; and in certain respects this seeming is well founded. But the logic of the theory of probabilities, which the kinetic theory uses in deducing the physical properties of gases from the statistical averages of collisions, and free paths of the hypothetical molecules, is no longer reducible to the logic of mechanics. For the velocity, the path, and the collision of each individual molecule are all indifferent facts for this kinetic theory of gases; which devotes itself to the study of probabilities and of tendencies. And its methods are in part those which the procedure of the insurance actuaries exemplifies. The logic in question is one which in some respects still needs further elucidation. For even up to the present time the logic of the theory of probabilities is a controverted topic. But there are a few features of the situation about which nobody who looks carefully into the subject can retain, I think, any serious doubt.

First, then, the average behavior of a very large collection of irregularly moving objects has characters which are decidedly lawful, even although the laws in question are what may be called laws of chance.

The recent familiar use of statistical diagrams for illustrative purposes has made this law of chance more familiar to many classes of students than it was in the day when Maxwell wrote certain words which you will find in his "Theory of Heat."[1] These words give you the very heart of the statistical aspect of nature.

> The distribution of the molecules according to their velocities is found to be of exactly the same mathematical form as the distribution of observations according to the magnitude of their errors, as described in the theory of errors of observation. . . . Whenever in physical phenomena some cause exists over which we have no control, and which produces a scattering of the particles of matter, a deviation of observations from the truth, or a diffusion of velocity or of heat, mathematical expressions of this exponential form are sure to make their appearance.

This, then, is in concrete form the law of random distribution, the form of iron necessity which one finds in the realm of chance.

All this law of chance variation was, of course, at that time no novelty, although the popular use of statistics has since made it more familiar. What was new, however, was the fact that when Maxwell computed the consequences which followed from supposing the existence of his swarm of colliding molecules with their chance distribution of velocities, he was able to deduce not only the principal physical properties of gases, but in particular those properties which, like all the phenomena which illustrate the second law of the theory of energy, are not expressible in terms of merely mechanical laws, unless these laws are applied to the case of a system complex enough to ensure that the velocities of its molecules shall approximate closely to this chance distribution.

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[1] Page 309 of the Appleton edition of 1875.

Since Maxwell's time, the same theoretical methods have been applied to a vast range of physical phenomena, with the general result that the second law of the theory of energy is now generally regarded, by all except the extreme Energetiker, as essentially a statistical law. So viewed, the second law of energy becomes a principle stated wholly in terms of the theory of probability. It is the law that the physical world tends, in each of its parts, to pass from certain less probable to certain more probable configurations of its moving particles. As thus stated the second principle not only becomes a law of evolution, an historical principle, but also ceases to be viewed as any mechanically demonstrable or fundamentally necessary law of nature. Whether nature is a mechanism or not, energy, according to the kinetic theory, runs down hill as it does for statistical and not for mechanical reasons. Energy need not always run down hill; and in fact would not do so if there were present in nature any persistent tendency, however imperfect, towards a suitable sorting of molecules. Maxwell suggested in his image of the demons sorting the atoms of a gas, how such a tendency might make energy run up hill instead of down, without the violation of any mechanical principle.

More recently Boltzmann, in his further development of Maxwell's hypothesis, pointed out how the theory of probability itself requires that, in the course of very vast intervals of time, there must occur some occasional concentrations of energy and some sensible unmixings--some reversals of the diffusion of gases, in case indeed the kinetic theories are themselves true. And still more recently Arrhenius has suggested that the nebulae may furnish the conditions for the occasional if not the general reversal of the second law of the theory of energy. Of such speculations I can of course form no judgment. They interest us here only as examples of the logic of the statistical view of nature.

In sum, all these investigations have tended to this general result: If a law of the physical world does not appear consistent with the mechanical view of nature so long as you confine your attention to a single system of bodies, whose individual movements you follow and compute, this law may still become perfectly intelligible when viewed as the expression of the average behavior of a kinetic system complex enough to give an opportunity for the application of statistical laws, and for the use of the conception of probability.

VIII. THE CANONICAL FORM OF SCIENTIFIC THEORIES

All the foregoing instances may appear to you merely to suggest that, in dealing with mechanisms too complicated to be the object of a direct computation, our ignorance may force us to make use of statistical modes of computation. These statistical methods, you may say, are convenient devices whereby we neutralize, for certain special purposes, the defects of our mechanical knowledge.

If the insurance actuaries--so you may say--could use a sufficient knowledge of the world's mechanism, they would compute the precise time when each individual man is to die, just as the astronomers compute the eclipses. An almanac of mortality would take the place of the present nautical almanac. Everybody's funeral would be announced, if that were convenient, years in advance; and life insurance would appear to be a blundering and an awkward substitute for scientific prediction. Because and only because, as a fact, no knowledge of the differential equations of the precise movements of matter, and no exact measurements of the accelerations or of the other rates of change in these movements gives us the power to predict the phenomena of nature in their detail, including the movements which determine life and death, we are obliged to substitute a statistical definition of the probable tendencies of a definable proportion of great numbers of men to die, in a way which varies with their numbers and their ages, for the precise knowledge of the hour of each man's death which we should all regard as a scientific ideal, if we could know the mechanism of life and death. The statistical view is a mere substitute for a mechanical view which our ignorance makes us unable to use, in the individual case, with sufficient accuracy. Such may be your comment. The nautical almanac (so you may say) is the model of what applied science ought to be. The mortality table is the convenient summary due to a necessary scientific evil. It is a device for recording our ignorance of the details of the world's mechanism along with our imperfect knowledge of certain probable and approximate tendencies to which the averages of many human lives are subject.

In other words, you may be disposed to insist: "Mechanical theories are the canonical forms towards which a growing scientific knowledge guides our way. Computations of individual events in terms of invariant laws whose validity is independent of time, are the models of what our scientific ideals seek. The statistical view of very complex mechanisms is an asylum in which our ignorance, perforce, has to find its refuge whenever, as in the case of the swarms of molecules and the labyrinthine complications of organisms, the mechanical view of nature, as applied by us, loses its way."

In answer to this very natural comment, I am next led to say that, whether the natural world is a mechanism or not, the statistical view of nature would be, and so far as we know the facts is, applicable to sufficiently complicated systems of things and events, not as a mere substitute for these more exact computations which our ignorance of mechanical laws makes necessary, but as an expression of a very positive, although only probable and approximate, knowledge, whose type all of the organic and social sciences, as well as most aspects of the inorganic sciences, illustrate. There is therefore good reason to say that not the mechanical but the statistical form is the canonical form of scientific theory, and that if we knew the natural world millions of times more widely and minutely than we do, the mortality tables and the computations based upon a knowledge of averages would express our scientific knowledge about individual events much better than the nautical almanac would do. For our mechanical theories are in their essence too exact for precise verification. They are verifiable only approximately. Hence, since they demand precise verification, we never know them to be literally true.

But statistical theories, just because they are deliberate approximations, are often as verifiable as their own logical structure permits. They often can be known to be literally, although only approximately, true.

This assertion is, in its very nature, a logical assertion. It is not any result of any special science, or of any one group of sciences. It solves no one problem about vitalism. It is a general comment on the value of the statistical point of view.

But, if the assertion is true, it tends to relieve us of a certain unnecessary reverence for the mechanical form of scientific theory--a reverence whose motives are neither rationally nor empirically well founded. It is the merit of Charles Peirce to have emphasized these logical considerations. Their importance for the study of scientific methods has grown greater with every year since 1891, when he began the publication of his remarkable papers in the Monist, entitled: "The Architecture of Theories," "The Doctrine of Necessity Examined" and "The Law of Mind." These papers are fragmentary; and yet in their way they are classical statements of the limitations of the mechanical view of nature, and of the significance of the statistical view of nature.

As I close, let me merely outline some aspects of Peirce's extension of the statistical view of nature beyond the range which Maxwell's and Boltzmann's study of the theory of gases directly exemplified.

IX. APPLICATIONS OF THE STATISTICAL VIEW TO THEORIES OF NON-MECHANICAL SYSTEMS. AGGREGATION AND ASSIMILATION AS STATISTICAL TENDENCIES.

It at first seems, I have said, as if the statistical methods of the kinetic theory were applicable only to mechanisms whose complications were too vast to make it possible to follow in individual detail their necessary sequences of movements.

But this seeming is unfounded. Let me summarize in my own words a few considerations which Peirce summarily states, and which, to my mind, get a constantly increasing importance as the statistical view of nature comes to be applied to wider and wider fields of research.

Suppose an aggregate of natural objects which contains a very great number of members, each one of which is subject to some more or less exhaustively definable range of possible variations. These objects may be things or events, at your pleasure. They may be molecules or stars or cells or multicellular organisms or members of a society or observations of a physical quantity or proposals of marriage or homicides or literary compositions or moral agents or whatever else you will. The essential basis which is needed for a statistical view of such an aggregate is this:

First, the members of each aggregate must actually form a collection which is, for some physical or moral reason, a genuine and therefore in some way a definable whole.

Next, some more or less systematic tendency towards a mutual assimilation of the fortunes, the characters or the mutual relations of the members of this aggregate must exist. This tendency toward mutual assimilation may be of very various sorts.

The policyholders of an insurance company tend to assimilate the fortunes of their various investments when they all of them pay their premiums to the same company. The stars tend to a certain assimilation of the mutual relations amongst those photographs of their various spectra which chance to get collected on the photographic plates of the same astronomical observatory. For, as a consequence of this aggregation of photographs, the stellar spectra in question may tend to be classified; and the logical, as well as the other socially important, and the physical fortunes of objects which are once viewed or arranged or tabulated as objects belonging to the same class, tend, in general, to a further mutual assimilation.

Birds of a feather not only flock together, but tend to get statistically similar fortunes, when they come into chance contact with other birds or with breeders, with hunters or with biometrical statisticians.

All objectively well-founded classification is not only founded upon real similarities amongst the objects which belong to an aggregate, but tend to some increase of these similarities, in so far as these objects are not changeless mathematical entities, but are natural objects, whose fortunes are subject to change.

One of the most widely applicable laws of nature is, in fact, the law, wholly indefinable in mechanical terms, but always expressible in terms of statistical tendencies--the law that aggregation tends to result in some further and increasing mutual assimilation of the members of the aggregate. This assimilation may express itself in the fact that one classification or aggregation leads both logically and physically to another and deeper and also wider aggregation.

If the stars are already physically classified into two distinct drifts, which move through each other in two different directions, and if the stars in question tend to get the photographs of their spectra assembled in the same observatory, then the classes into which the photographs tend in the long run to be grouped also tend to be such that, at least for some one resulting classification or aggregation of the photographs, the photographs of the spectra of the stars of one of the star drifts are grouped together, not only in the ideas which the astronomers form, but in the physical arrangements towards which certain groups of photographs, of symbols and of statistical tables, persistently tend.

The principles here involved depend upon the sorts of assimilation which the radiant phenomena of light make possible. For a photograph is a physical expression of a certain tendency whereby the structure of a photographic plate tends to be assimilated to the molecular structure and state of a radiating object--say a star. When the photographs of stellar spectra are grouped in classes, a secondary assimilation tends to take place, since similar spectra tend to get either placed or tabulated in similar ways. When this secondary assimilation of the photographs leads to an indirect discovery of the existence of the two star drifts themselves, a tertiary assimilation of the fortunes of those stars whose proper motions are sufficiently similar takes place, and tends to get represented in the knowledge of different astronomers.

The ideas of these various astronomers tend to further assimilation through the means used in scientific communication. The radiation of scientific knowledge continues the natural process which the radiation of light and the making of photographs of stellar spectra have already illustrated, and the rule continues to be illustrated that mutual assimilation is one aspect of classification and aggregation, and is a cumulative statistical tendency which accompanies them both.

The insurance companies and the transformation of modern civilization through the extension and aggregation of modes and devices whereby insurance is accomplished, furnish numerous other examples of this law of the fecundity of aggregation. The law, as I have said, holds in general for non-mechanical systems, although, as stellar evolution seems to indicate, it can also hold for mechanical systems. It may hold, in fact, for all natural processes which involve evolution.

Clerk Maxwell himself believed that the sharp distinction which separate the different classes of elementary atoms, and the different types of molecular structure which determine the spectra of the molecules of different elements, are signs that no kinetic theory of the evolution of the chemical elements would ever be possible. It is precisely here that the latest advances, on the still so imperfectly defined outlying boundaries of physical and of chemical research, give a new significance to the statistical view of nature, by showing that if we take account of sufficiently large aggregates of things and of events, a kinetic theory of the evolution of chemical elements becomes a possibility worthy of future investigation, and certain to receive, in connection with the phenomena of radio-activity, further investigation upon statistical lines, whatever be the further fortunes of the mechanical view of nature, or of this problem about the evolution of the elements.

Of such speculations one can say that, if ever a theory of the evolution of the chemical elements becomes feasible, it will be, in part at least, a statistical theory, and will illustrate in new ways how widespread in material nature is the tendency to that mutual assimilation which all the phenomena of radiant energy illustrate, and of which the relatively uniform constitution and distribution of each one of the various chemical elements through vast ranges of the physical universe may well be the result.

In brief, the evolution of stars, of elements, of social orders, of minds and of moral processes, apparently illustrates the statistical fecundity of nature's principal tendency--the tendency to that mutual assimilation which both defines aggregates, that is, real classes of natural objects, and tends to keep these classes or aggregates permanent in the world and to increase both their wealth of constitution and their extent.

Now it is this principle of the fecundity of aggregation which seems to be the natural expression, in statistical terms, for the tendency of nature towards what seems to be a sort of unconscious teleology--towards a purposiveness whose precise outcome no finite being seems precisely to intend. It is a statistically definable rule that changeable aggregates, when they are real at all, result from likenesses which their very existence tends both to increase and to diversify. The social fecundity of the principle of insurance illustrates this natural tendency. That marvelous result of the aggregation of scientific observers, of tabulations and of photographs, of the radiant phenomena which makes the stars visible and of the microscopic phenomena and the logical interests which make probability definable--that marvelous result of these various aggregations which constitutes the whole procedure and outcome of modern inductive science itself, is an expression of this same general tendency--apparently the most vital and the most vitalizing tendency both of the physical and of the spiritual world--the tendency of aggregation and of classification to be fruitful both of new aggregations and of the orderly array of natural classes and of natural laws.

In the purely logical and mathematical worlds this tendency can get, and does get, precise description in terms of the pure logic of number and of order. In the physical world, in the world of time and change, this principle gets further expressed as a statistical rather than a mechanical law--the law that classes, aggregations and organizations tend towards a definable sort of evolution.

As Charles Peirce pointed out, you need not suppose the real world to be mechanical in order to define and to conceive this sort of evolution. You need only suppose (1) the presence of the just-mentioned tendency to form aggregates, and of the mutual assimilation of the various parts of nature; (2) the statistically definable tendency to some sort of sorting or selection of the probable results to which any definable average constitution of the natural world at any moment leads; and (3) a tendency--and once more, a statistical and non-mechanical tendency, towards a formation of habits, and towards a repetition of such types of movement as have once appeared. Suppose these three tendencies (aggregation, selection and habit--and the statistical method shows these three to be widespread in the physical world); suppose these three, and you can define a process of evolution, never mechanical and never merely expressive of any previously settled designs, either of gods or of men. This process of evolution will then lead from mere chance towards the similation of mechanism, from disorderly to a more orderly arrangement, not only of things and of individual events, but of the statistically definable laws of nature; that is., of the habits which nature gathers as she matures. The philosophy of nature which will result will show how nature may well tend to appear in certain aspects more and more teleological, and to manifest what Greek vitalism found in nature. Whether the whole world is ultimately and consciously teleological or not, this view of nature would of course be unable to decide. But it would lay stress upon the thought that what is indeed most vital about the world is that which also characterizes the highest life of the spirit, namely, the fecundity of whatever unites either electrons or souls or stars into streams or into other aggregations that, amid all chances, illustrate some tendency to orderly cooperation.

If this view of nature has any foundation, gentlemen, then, as the whole progress of inductive science illustrates, the way to further such scientific evolution is to get together, and to leave the rest to the statistically definable tendencies of nature. These are tendencies away from the chance distributions which the bell-shaped curve of random distribution illustrates, towards the orderliness of which the mechanical view of nature gives us one illustration, and by no means the most probably true illustration.

I should suppose, then, that whatever notes you may compare in these meetings, you will probably frequently and variously illustrate the statistical view of nature. This view is ill understood by those who think only how dry statistical tables and averages may seem. Mechanism is rigid, but probably never exactly realized in nature. But life, although it has its history, has also its statistics. And averages cease to be dry when they are averages that express the unities and the mutual assimilations in which the common ideals and interests, the common hopes and destinies of the men, of the social orders, of the deeds--yes, and perhaps of the stars and of all the spiritual world are bound up and are expressed.

Do you wish to experiment upon some new processes of social aggregation, of mutual assimilation, and of the study of photographs of your various spiritual spectra?

This practical question is for you to consider.

Chapter III

THE PROBLEM OF TRUTH IN THE LIGHT OF RECENT DISCUSSION[1]

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The question: What is Truth? is a typical philosophical problem. But it has been by no means at all times equally prominent throughout the history of philosophy. The ages in which it has come to the front have been those wherein, as at present, a keenly critical spirit has been predominant. At such times metaphysical interests are more or less subordinated, for a while, to the problems about method, to logical researches, or to the investigations which constitute a Theory of Knowledge.

Such periods, as we know, have recurred more than once since scholastic philosophy declined. And such a period was that which Kant dominated. But the sort of inquiry into the nature of truth which Kant's doctrine initiated quickly led, at the close of the eighteenth century, to a renewed passion for metaphysical construction. The problem regarding the nature of truth still occupied a very notable place in the doctrine of Fichte. It constituted one of the principal concerns, also, of Hegel's so much neglected and ill-understood "Phänomenologie des Geistes." And yet both in the minds of the contemporaries of Fichte and of Hegel, and still more in those of their later disciples and opponents, the problem of truth went again into the background when compared with the metaphysical, the

[1] An address delivered before the International Congress of Philosophy at Heidelberg, in September, 1908.

ethical, and the theological interests which constructive idealism and its opponents, in those days, came to represent. Hence wherever one looks, in the history of philosophical opinion between 1830 and 1870, one sees how the problem of truth, although never wholly neglected, still remained, for some decades, out of the focus of philosophical interest.

But the scene rapidly changed about and after the year 1870. Both the new psychology and the new logic, which then began to flourish, seemed, erelong, almost equally to emphasize the importance of a reconsideration of the problem as to the nature of truth. These doctrines did this, especially because the question whether logic was henceforth to be viewed as a part of psychology became once more prominent, so soon as the psychological researches then undertaken had attracted the strong interest of the philosophical public. And meanwhile the revived interest in Kant, growing, as it did, side by side with the new psychology, called for a reinterpretation of the problems of the critical philosophy. The reawakening of Idealism, in England and in America, called attention, in its own way, to the same problem. The modern philosophical movement in France,--a movement which was, from the outset, almost equally made up of a devotion to the new psychology and of an interest in the philosophy of the sciences, has coöperated in insisting upon the need of a revision of the theory of truth. And to complete the story of the latest philosophy, recent tendencies in ethics, emphasizing as they have done the problems of individualism, and demanding a far-reaching reconsideration of the whole nature of moral truth, have added the weight of their own, often passionate, interest to the requirements which are here in question.

The total result is that we are just now in the storm and stress of a reëxamination of the whole problem of truth. About this problem the philosophical interest of to-day centers. Consequently, whether you discuss the philosophy of Nietzsche or of mathematics,--whether the Umwertung aller Werte or the "class of all classes,"--whether Mr. Russell's "Contradiction" or the Uebermensch is in question,--or whether none of these things attract you at all, so that your inquiries relate to psychology, or to evolution, or to the concepts of the historical sciences, or to whatever other region of philosophy you please,--always the same general issue has sooner or later to be faced. You are involved in some phase of the problem about the nature of truth.

So much, then, as a bare indication of the historical process which has led us into our present position. I propose, in the present address, to offer an interpretation of some of the lessons that, as I think, we may learn from the recent discussions of the problems whose place in all our minds I have thus indicated.

I.

It seems natural to begin such a discussion by a classification of the main motives which are represented by the principal recent theories regarding the nature of truth. In enumerating these motives I need not dwell, in this company, upon those historical inferences and traditions whose presence in recent thought is most easily and universally recognized. That Empiricism,--due to the whole history of the English school, modified in its later expressions by the Positivism of a former generation, and by the types of Naturalism which have resulted from the recent progress of the special sciences,--that, I say, such empiricism has affected our modern discussion of the nature of truth,--this we all recognize. I need not insist upon this fact. Moreover, the place which Kant occupies in the history of the theory of truth,--that again is something which it is needless here to emphasize. And that the teaching of Fichte and of Hegel, as well as still other idealistic traditions, are also variously represented by present phases of opinion regarding our problem, we shall not now have to rehearse. I presuppose, then, these historical commonplaces. It is not, however, in terms of these that I shall now try to classify the motives to which the latest theories of truth are due.

These recent motives, viewed apart from those unquestionably real influences of the older traditions of the history of philosophy are, to my mind, three in number:

First, there is the motive especially suggested to us modern men by the study of the history of institutions, by our whole interest in what are called evolutionary processes, and by a large part of our recent psychological investigation. This is the motive which leads many of us to describe human life altogether as a more or less progressive adjustment to a natural environment. This motive incites us, therefore, to judge all human products and all human activities as instruments for the preservation and enrichment of man's natural existence. Of late this motive, whose modern forms are extremely familiar, has directly affected the theory of truth. The result appears in a part, although not in the whole, of what the doctrines known as Instrumentalism, Humanism, and Pragmatism have been of late so vigorously teaching, in England, in America, in Italy, in France, and, in still other forms, in Germany.

From the point of view which this motive suggests, human opinions, judgments, ideas, are part of the effort of a live creature to adapt himself to his natural world. Ideas and beliefs are, in a word, organic functions. And truth, in so far as we men can recognize truth at all, is a certain value belonging to such ideas. But this value itself is simply like the value which any natural organic function possesses. Ideas and opinions are instruments whose use lies in the fact that, if they are the right ones, they preserve life and render life stable. Their existence is due to the same natural causes that are represented in our whole organic evolution. Accordingly, assertions or ideas are true in proportion as they accomplish this their biological and psychological function. The value of truth is itself a biological and psychological value. The true ideas are the ones which adapt us for life as human beings. Truth, therefore, grows with our growth, changes with our needs, and is to be estimated in accordance with our success. The result is that all truth is as relative as it is instrumental, as human as it is useful.

The motive which recent Instrumentalism or Pragmatism expresses, in so far as it takes this view of the nature of truth, is of course in one sense an ancient motive. Every cultivated nation, upon beginning to think, recognizes in some measure such a motive. The Greeks knew this motive, and deliberately connected both the pursuit and the estimate of truth with the art of life in ways whose problematic aspects the Sophists already illustrated. Socrates and his followers, and later the Stoics as well as the Epicureans, also considered, in their various ways, this instrumental aspect of the nature of truth. And even in the Hindoo Upanishads one can find instances of such humanistic motives influencing the inquiry into the problem of truth. But it is true that the historical science of the nineteenth century, beginning, as it did, with its elaborate study of the history of institutions, and culminating in the general doctrines regarding evolution, has given to this motive an importance and a conscious definiteness such as makes its recent embodiment in Pragmatism a very modern and, in many ways, a novel doctrine about the nature of truth.

II.

But closely bound up with this first motive in our recent thinking there is a second motive, which in several ways very strongly contrasts with the first. Yet in many minds these two motives are so interwoven that the writers in question are unaware which motive they are following when they utter their views about the nature of truth. No doubt one may indeed recognize the contrast between these motives, and may, nevertheless, urge good reasons for following in some measure both of them, each in its own way. Yet whoever blindly confuses them is inevitably led into hopeless contradictions. As a fact, a large number of our recent pragmatists have never learned consciously to distinguish them.

Yet they are indeed easy to distinguish, however hard it may be to see how to bring them into a just synthesis.

This second motive is the same as that which, in ethics, is responsible for so many sorts of recent Individualism. It is the motive which in the practical realm Nietzsche glorified. It is the longing to be self-possessed and inwardly free, the determination to submit to no merely external authority. I need not pause to dwell upon the fact that, in its application to the theory of truth, precisely as in its well-known applications to ethics, this motive is Protean. Every one of us is, I suppose, more or less under its influence.

Sometimes, this motive appears mainly as a skeptical motive. Then it criticizes, destructively, traditional truth and thereupon leaves us empty of all assurances. But sometimes it assumes the shape of a sovereign sort of rationalism, whereby the thinking subject, first rebelling against outer authority, creates his own laws, but then insists that all others shall obey these laws. In other cases, however, it takes the form of a purely subjective idealism, confident of its own but claiming no authority. Or again, with still different results, it consciously unites its ethical with its theoretical interests, calls itself "Personal Idealism," and regards as its main purpose, not only the freeing of the individual from all spiritual bondage, theoretical and practical, but also the winning for him of an inner harmony of life. In general, in its highest as in some of its less successful embodiments, when it considers the sort of truth that we ought most to pursue, this motive dwells, as Professor Eucken has so effectively taught it to dwell, upon the importance of a <u>Lebensanschauung</u> as against the rigidity and the pretended finality of a mere <u>Weltanschauung</u>.

But meanwhile, upon occasion, this same motive embodies itself in various tendencies of the sort known as Irrationalism. In this last case, it points out to us how the intelligence, after all, is but a single and a very narrow function of our nature, which must not be allowed to supersede or even too much to dominate the rest of our complex and essentially obscure, if fascinating, life. Perhaps, on the very highest levels of life, as it hereupon suggests to us: Gefühl ist alles. If not, then at all events, we have the alternative formula: Im Anfang war die Tat. Or, once again, the solving word of the theory of truth is Voluntarism. Truth is won by willing, by creative activities. The doer, or perhaps the deed, not only finds, but is, the truth. Truth is not to be copied, but to be created. It is living truth. And life is action.

I have thus attempted to indicate, by well-known phrases, the nature of this second motive,--one whose presence in our recent theories of truth I believe that you will all recognize. Despite the Protean character and (as you will all at once see) the mutually conflicting characters of its expressions, you will observe, I think, its deeper unity, and also its importance as an influence in our age. With us at present it acts as a sort of ferment, and also as an endless source of new enterprises. It awakens us to resist the most various kinds of doctrinal authority,--scientific, clerical, academic, popular. It inspires countless forms of Modernism, both within and without the boundaries of the various confessions of Christendom. As an effective motive, one finds it upon the lowest as also upon the highest levels of our intellectual and moral life. In some sense, as I have said, we all share it. It is the most characteristic and the most problematic of the motives of the modern world. Anarchism often appeals to it; yet the most saintly form of devotion, the most serious efforts for the good of mankind, and our sternest and loftiest spiritual leaders, agree in employing it, and in regarding it as in some sense sacred.

Our age shares this motive with the age of the French Revolution, of the older Idealistic movement, and of the Romantic School. All the more unfortunate, as I think, is the fact that many who glory in the originality of their own recent opinions about the nature of truth, know so little of the earlier history of this motive, read so seldom the lesson of the past, and are thus so ill-prepared to appreciate both the spiritual dignity and the pathetic paradox of this tendency to make the whole problem of truth identical with the problem of the rights and the freedom of the individual.

III.

I turn herewith to the third of the motives that I have to enumerate. In its most general form it is a very ancient and familiar motive. It is, indeed, very different from both of the foregoing. Superficially regarded, it seems at first sight, less an expression of interests that appear ethical. At heart, however, it is quite as deep a motive as either of the others, and it is in fact a profoundly ethical motive as well as a genuinely intellectual one. One may say that, in a sense and to some degree, it pervades the whole modern scientific movement, is present wherever two or three are gathered together for a serious exchange of scientific opinions, and is, in most cases, the one motive that, in scientific assemblies, is more or less consciously in mind whenever somebody present chances to refer to the love of truth, or to the scientific conscience of his hearers.

I have called this third on our list of motives an ancient motive. It is so. Yet in modern times it has assumed very novel forms, and has led to scientific and, in the end, to philosophical enterprises which, until recently, nobody would have thought possible.

It would be unwise at this point to attempt to define this motive in abstract terms. I must first exemplify it. When I say that it is the motive to which the very existence of the exact sciences is due, and when I add the remark that our scientific common sense knows this motive as the fondness for dispassionately weighing evidence, and often simply names it the love of objectivity, I raise more questions in your minds regarding the nature of this motive than at this point I can answer. If, however, anybody suggests, say from the side of some form of recent pragmatism, that I must be referring to the nowadays so deeply discredited motives of a pure "Intellectualism," I repudiate at once the suggestion. The motive to which I refer is intensely practical. Men have lived and died for it, and have found it inestimably precious. I know of no motive purer or sweeter in human life. Meanwhile, it indeed chances to be the motive which has partially embodied itself in Pure Mathematics. And neither the tribe of Nietzche nor the kindred of the instrumentalists have been able justly to define it.

What I am just now interested to point out is that this motive has entered, in very novel ways, into the formulation of certain modern theories of truth. And when I speak of its most novel forms of expression, the historical process to which I refer is the development of the modern critical study of the foundations of mathematics.

To philosophical students in general the existence of metageometrical researches, which began at the outset of the nineteenth century, has now been made fairly familiar. But the non-Euclidean geometry is but a small fragment of that investigation of the foundations of mathematical truth which went on so rapidly during the nineteenth century. Among the most important of the achievements of the century in this direction were the new definitions of continuity and the irrational numbers, the modern exact theory of limits, and the still infant theory of Assemblages. Most important of all, to my mind, were certain discoveries in the field of Logic of which I shall later say a word. I mention these matters here as examples of the influence of a motive whose highly technical applications may make it seem to one at a distance hopelessly intellectualistic, but whose relation to the theory of truth is close, just because, as I think, its relation to truly ethical motives is also extremely intimate.

The motive in question showed itself at the outset of the nineteenth century, and later in the form of an increased conscientiousness regarding what should be henceforth accepted as a rigid proof in the exact sciences. The Greek geometers long ago invented the conception of rigid methods of proof and brought their own methods, in certain cases, very near to perfection. But the methods that they used proved to be inapplicable to many of the problems of modern mathematics. The result was that, in the seventeenth and eighteenth centuries, the mathematical sciences rapidly took possession of new realms of truth, but in doing so sacrificed much of the old classic rigidity. Nevertheless, regarded as the instrumentalists now desire us to regard truth, the mathematical methods of the eighteenth century were indeed incomparably more successful in adjusting the work of the physical sciences to the demands of experience than the methods of the Greek geometers had ever been. If instrumentalism had been the whole story of man's interest in truth, the later developments would have been impossible. Nevertheless the modern scientific conscience somehow became increasingly dissatisfied with its new mathematical possessions. It regarded them as imperfectly won. It undertook to question, in a thousand ways, its own methods and its own presuppositions. It learned to reject altogether methods of proof which, for a time, had satisfied the greatest constructive geniuses of earlier modern mathematics. The result has been the development of profoundly novel methods, both of research and of instruction in the exact sciences. These methods have in many ways brought to a still higher perfection the Greek ideal of rigid proof. Yet the same methods have shown themselves to be no mere expressions of a pedantic intellectualism. They have meant clearness, self-possession, and a raising of the scientific conscience to higher levels. Meanwhile, they proved potent both in conquering new realms and in discovering the wonderful connections that we now find linking together types of exact truth which at first sight appeared to be hopelessly diverse.

In close union with the development of these new methods in the exact sciences, and, as I may say, in equally close union with this new scientific conscience, there has gradually come into being a reformed Logic, --a logic still very imperfectly expounded in even the best modern textbooks, and as yet hardly grasped, in its unity, by any one investigator,--but a logic which is rapidly progressing, which is full of beauty, and which is destined, I believe, profoundly to influence, in the near future, our whole philosophy of truth. This new logic appears to offer to us an endless realm for detailed researches. As a set of investigations it is as progressive as any instrumentalist can desire. The best names for it, I think, are the names employed by several different thinkers who have contributed to its growth. Our American logician, Mr. Charles Peirce, named it, years ago, the Logic of Relatives. Mr. Russell has called it the Logic, or the Calculus, of Relations. Mr. Kempe has proposed to entitle it the Theory of Mathematical Form. One might also call it a new and general theory of the Categories. Seen from a distance, as I just said, it appears to be a collection of highly technical special researches, interesting only to a few. But when one comes into closer contact with any one of its serious researches, one sees that its main motive is such as to interest every truthful and reflective inquirer who really grasps that motive, while the conception of truth which it forces upon our attention is a conception which neither of the other motives just characterized can be said adequately to express.

In so far as the new logic has up to this time given shape to philosophical theories of truth, it in part appears to tend towards what the pragmatists nowadays denounce as Intellectualism. As a fact Mr. Bertrand Russell, the brilliant and productive leader of this movement in England, and his philosophical friend Mr. George Moore, seem to regard their own researches as founded upon a sort of new Realism, which views truth as a realm wholly independent of the constructive activities by which we ourselves find or pursue truth. But the fact that Mr. Charles Peirce, one of the most inventive of the creators of the new logic, is also viewed by the Pragmatists as the founder of their own method, shows how the relation of the new logic to the theory of truth is something that still needs to be made clear. As a fact, I believe that the outcome of the new logic will be a new synthesis of Voluntarism and Absolutism.

What I just now emphasize is, that this modern revision of the concepts of the exact sciences, and this creation of a new logic, are in any case due to a motive which is at once theoretical and ethical. It is a motive which has defined standards of rigidity in proof such as were, until recently, unknown. In this sense it has meant a deepening and quickening of the scientific conscience. It has also seemed, in so far, to involve a rejection of that love of expediency in thinking which is now a favorite watchword of pragmatists and instrumentalists. And when viewed from this side the new logic obviously tends to emphasize some form of absolutism, to reject relativism in thinking, to make sterner requirements upon our love of truth than can be expressed in terms of instrumentalism or of individualism. And yet the motive which lies beneath this whole movement has been, I insist, no barren intellectualism. The novelty of the constructions to which this motive has led,--the break with tradition which the new geometry (for instance) has involved,--such things have even attracted, from a distance, the attention of some of the least exactly trained of the pragmatist thinkers, and have aroused their hasty and uncomprehending sympathy. "This non-Euclidean geometry," they have said, "these novel postulates, these '<u>freie Schöpfungen des menschlichen Geistes</u>' (as Dedekind, himself one of the great creative minds of the new logical movement, has called the numbers),--well, surely these must be instances in favor of our theory of truth. Thus, as we should have predicted, novelties appear in what was supposed to be an absolutely fixed region. Thus (as Professor James words the matter), human thought 'boils over,' and ancient truths alter, grow, or decay." Yet when modern pragmatists and relationists use such expressions, they fail to comprehend the fact that the new discoveries in these logical and mathematical fields simply exemplify a more rigid concept of truth than ever, before the new movement began, had been defined in the minds of the mathematicians themselves. The non-Euclidean geometry, strange to say, is not a discovery that we are any freer than we were before to think as we like regarding the system of geometrical truth. It is one part only of what Hilbert has called the "logical analysis" of our concept of space. When we take this analysis as a whole, it involves a deeper insight than Euclid could possibly possess into the unchangeable necessities which bind together the system of logical relationships that the space of our experience merely exemplifies. Nothing could be more fixed than are these necessities. As for the numbers, which Dedekind called "freie Schöpfungen,"--well, his own masterpiece of logical theory is a discovery and a rigid demonstration of a very remarkable and thoroughly objective truth about the fundamental relations in terms of which we all of us do our thinking. His proof that all of the endless wealth of the properties of the ordinal numbers follows from a certain synthesis of two of the simplest of our logical conceptions, neither one of which, when taken alone, seems to have anything to do with the conception of order or of number,--this proof, I say, is a direct contribution to a systematic theory of the categories, and, as such, is, to the logical inquirer, a dramatically surprising discovery of a realm of objective truth, which nobody is free to construct or to abandon at his pleasure. If this be relativism, it is the relativism of an eternal system of relations. If this be freedom, it is the divine freedom of a self-determined, but, for that very reason, absolutely necessary fasion of thought and of activity.

Well,--to sum up,--this third motive in modern inquiry has already led us to the discovery of what are, for us, novel truths regarding the fundamental relations upon which all of our thought and all of our activity rest. These newly discovered truths possess an absoluteness which simply sets at naught the empty trivialities of current relativism. Such truth has, in fact, the same sort of relation to the biologically "instrumental" value of our thinking processes as the Theory of Numbers (that "divine science," as Gauss called it) has to the account books of the shopkeeper.

And yet, as I must insist, the motive that has led us to this type of absolutism is no pure intellectualism. And the truth in question is as much a truth about our modes of activity as the purest voluntarism could desire it to be. In brief, there is, I believe, an absolute voluntarism, a theory of the way in which activities must go on if they go on at all. And, as I believe, just such a theory is that which in future is to solve for us the problem of the nature of truth.

I have illustrated our third motive at length. Shall I now try to name it? Well, I should say that it is at bottom the same motive that lay at the basis of Kant's Critical Philosophy; but it is this motive altered by the influence of the modern spirit. It is the motive which leads us to seek for clear and exact self-consciousness regarding the principles both of our belief and of our conduct. This motive leads us to be content only in case we can indeed find principles of knowledge and of action,--principles, not mere transient expediences, and not mere caprices. On the other hand, this motive bids us decline to accept mere authority regarding our principles. It requires of us freedom along with insight, exactness side by side with assurance, and self-criticism as well as search for the ultimate.

IV.

In thus sketching for you these three motives, I have been obliged to suggest my estimate of their significance. But this estimate has so far been wholly fragmentary. Let me next indicate the sense in which

I believe that each of these three motives tends, in a very important sense, to throw light upon the genuine theory of truth.

I begin here with the first of the three motives,--namely, with the motive embodied in recent instrumentalism. Instrumentalism views truth as simply the value belonging to certain ideas in so far as these ideas are biological functions of our organisms, and psychological functions whereby we direct our choices and attain our successes.

Wide and manifold are the inductive evidences which the partisans of such theories of truth adduce in support of their theory. There is the evidence of introspection and of the modern psychological theory of the understanding. Opinions, beliefs, ideas,--what are they all but accompaniments of the motor processes whereby, as a fact, our organisms are adjusted to their environment? To discover the truth of an idea, what is that for any one of us but to observe our success in our adjustment to our situation? Knowledge is power. Common sense long ago noted this fact. Empiricism has also since taught us that we deal only with objects of experience. The new instrumentalism adds to the old empiricism simply the remark that we possess truth in so far as we learn how to control these objects of experience. And to this more direct evidence for the instrumental theory of truth is added the evidence derived from the whole work of the modern sciences. In what sense are scientific hypotheses and theories found to be true? Only in this sense, says the instrumentalist, --only in this sense, that through these hypotheses we acquire constantly new sorts of control over the course of our experience. If we turn from scientific to moral truth, we find a similar result. The moral ideas of any social order are practical plans and practical demands in terms of which this social order endeavors, by controlling the activities of its members, to win general peace and prosperity. The truth of moral ideas lies solely in this their empirical value in adjusting individual activities to social demands, and in thus winning general success for all concerned.

Such are mere hints of the evidences that can be massed to illustrate the view that the truth of ideas is actually tested, and is to be tested, by their experienced workings, by their usefulness in enabling man to control his empirically given situation. If this be the case, then truth is always relative to the men concerned, to their experience, and to their situations. Truth grows, changes, and refuses to be tested by absolute standards. It happens to ideas, in so far as they work. It belongs to them when one views them as instruments to an end. The result of all this is a relativistic, an evolutionary, theory of truth. For such a view logic is a part of psychology,--a series of comments upon certain common characteristics of usefully working ideas and opinions. Ethical theory is a branch of evolutionary sociology. And in general, if you want to test the truth of ideas and opinions, you must look forward to their workings, not backward to the principles from which they might be supposed to follow, nor yet upwards to any absolute standards which may be supposed to guide them, and least of all to any realm of fixed facts that they are supposed to be required, willy nilly, to copy. Truth is no barren repetition of a dead reality, but belongs, as a quality, to the successful deeds by which we produce for ourselves the empirical realities that we want.

Such is the sort of evidence which my friends, Professor James and Professor Dewey, and their numerous followers, in recent discussion, have advanced in favor of this instrumental, practical, and evolutionary theory of truth. Such are the considerations which, in other forms, Mach has illustrated by means of his history and analyses of the work of modern science.

Our present comment upon this theory must be given in a word. It contains indeed a report of the truth about our actual human life, and about the sense in which we all seek and test and strive for truth, precisely in so far as truth-seeking is indeed a part of our present organic activities. But the sense in which this theory is thus indeed a true account of a vast range of the phenomena of human life is not reducible to the sense which the theory itself ascribes to the term "truth."

For suppose I say, reporting the facts of the history of science: "Newton's theory of gravitation proved to be true, and its truth lay in this: The definition and the original testing of the theory consisted in a series of the organic and psychological functions of the live creature Newton. His theories were for him true in so far as, after hard work, to be sure, and long waiting, they enabled him to control and to predict certain of his own experiences of the facts of nature. The same theories are still true for us because they have successfully guided, and still guide, certain observations and experiences of the men of to-day." This statement reduces the truth of Newton's theory to the type of truth which instrumentalism demands. But in what sense is my account of this matter itself a true account of the facts of human life? Newton is dead. As mortal man he succeeds no longer. His ideas, as psychological functions, died with him. His earthly experiences ceased when death shut his eyes. Wherein consists to-day, then, the historical truth that Newton ever existed at all, or that the countless other men whom his theories are said to have guided ever lived, or experienced, or succeeded? And if I speak of the men of to-day, in what sense is the statement true that they now live, or have experience, or use Newton's theory, or succeed with it as an instrument? No doubt all these historical and socially significant statements of mine are indeed substantially true. But does their truth consist in my success in using the ideal instruments that I use when I utter these assertions? Evidently I mean, by calling these my own assertions true, much more than I can interpret in terms of my experience of their success in guiding my act.

In brief, the truth that historical events ever happened at all; the truth that there ever was a past time, or that there ever will be a future time; the truth that anybody ever succeeds, except in so far as I myself, just now, in the use of these my present instruments for the transient control of my passing experience chance to succeed; the truth that there is any extended course of human experience at all, or any permanence, or any long-lasting success,--well, all such truths, they are indeed true, but their truth cannot possibly consist in the instrumental value which any man ever experiences as belonging to any of his own personal ideas or acts. Nor can this truth consist in anything that even a thousand or a million men can separately experience, each as the success of his own ideal instruments. For no one man experiences the success of any man but himself, or of any instruments but his own; and the truth, say, of Newton's theory consists, by hypothesis, in the perfectly objective fact that generations of men have really succeeded in guiding their experience by this theory. But that this is the fact no man, as an individual man, ever has experienced or will experience under human conditions.

When an instrumentalist, then, gives to us his account of the empirical truth that men obtain through using their ideas as instruments to guide and to control their own experience, his account of human organic and psychological functions may be,--yes, is,--as far as it goes, true. But if it is true at all, then it is true as an account of the characters actually common to the experience of a vast number of men. It is true, if at all, as a report of the objective constitution of a certain totality of facts which we call human experience. It is, then, true in a sense which no man can ever test by the empirical success of his own ideas as his means of controlling his own experiences. Therefore the truth which we must ascribe to instrumentalism, if we regard it as a true doctrine at all, is precisely a truth, not in so far as instrumentalism is itself an instrument for helping on this man's or that man's way of controlling his experience. If instrumentalism is true, it is true as a report of facts about the general course of history, of evolution, and of human experience,

--facts which transcend every individual man's experience, verifications, and successes. To make its truth consist in the mere sum of the various individual successes is equally vain, unless indeed that sum is a fact. But no individual man ever experiences that fact.

Instrumentalism, consequently, expresses no motive which by itself alone is adequate to constitute any theory of truth. And yet, as I have pointed out, I doubt not that instrumentalism gives such a substantially true account of man's natural functions as a truth seeker. Only the sense in which instrumentalism is a true account of human life is opposed to the adequancy of its own definition of truth. The first of our three motives is, therefore, useful only if we can bring it into synthesis with other motives. In fact it is useless to talk of the success of the human spirit in its efforts to win control over experience, unless there is indeed a human spirit which is more than any man's transient consciousness of his own efforts, and unless there is an unity of experience, an unity objective, real, and supratemporal in its significance.

V.

Our result so far is that man indeed uses his ideas as means of controlling his experience, and that truth involves such control, but that truth cannot be defined solely in terms of our personal experience of our own success in obtaining this control.

Hereupon the second of the motives which we have found influencing the recent theories of truth comes to our aid. If instrumentalism needs a supplement, where are we, the individual thinkers, to look for that supplement, except in those inner personal grounds which incline each of us to make his own best interpretation of life precisely as he can, in accordance with his own will to succeed, and in accordance with his individual needs?

To be sure, as one may still insist, we are always dealing with live human experience, and with its endless constraints and limitations. And when we accept or reject opinions, we do so because, at the time, these opinions seem to us to promise a future empirical "working," a successful "control" over experience,--in brief, a success such as appeals to live human beings. Instrumentalism in so far correctly defines the nature which truth possesses in so far as we ever actually verify truth. And of course we always believe as we do because we are subject to the constraint of our present experience. But since we are social beings, and beings with countless and varied intelligent needs, we constantly define and accept as valid very numerous ideas and opinions whose truth we do not hope personally to verify. Our act in accepting such unverified truths is (as Professor James states the case) essentially similar to the act of the banker in accepting credit values instead of cash. A note or other evidence of value is good if it <u>can</u> be turned into cash at some agreed time, or under specified conditions. Just so, an idea is true, not merely at the moment when it enables somebody to control his own experience. It is true if, under definable conditions which, as a fact, you or I may never verify, it <u>would</u> enable some human being whose purposes agree with ours to control his own experience. If we personally do not verify a given idea, we can still accept it then upon its credit value. We can accept it precisely as paper, which cannot now be cashed, is accepted by one who regards that paper as, for a given purpose, or to a given extent, equivalent to cash. A bond, issued by a government, may promise payment after fifty years. The banker may to-day accept such a bond as good, and may pay cash for it, although he feels sure that he personally will never live to see the principal repaid by the borrower.

Now, as Professor James would say, it is in this sense that our ideas about past time, and about the content of other men's minds, and about the vast psysical world, "with all its stars and milky ways," are accepted as true. Such ideas have for us credit values.

We accept these ideas as true because we need to trade on credits. Borrowed truth is as valuable in the spiritual realm as borrowed money is in the commercial realm. To believe a now unverified truth is simply to say: "I accept that idea, upon credit, as equivalent to the cash payments in terms of live experience which, as I assert, I could get in case I had the opportunity."

And so much it is indeed easy to make out about countless assertions which we all accept. They are assertions about experience, but not about our present experience. They are made under various constraints of convention, habit, desire, and private conviction, but they are opinions whose truth is for us dependent upon our personal assent and acquiescence.

Herewith, however, we face what is, for more than one modern theory of truth, a very critical question. Apparently it is one thing to say: "I accept this opinion upon credit," and quite another thing to say: "The truth of this opinion consists, solely and essentially, in the fact that it is credited by me." In seeming, at least, it is one thing to assert: "We trade upon credit; we deal in credits," and quite another thing to say: "There is no value behind this bond or behind this bit of irredeemable paper currency, except its credit value." But perhaps a modern theory of truth may decline to accept such a difference as ultimate. Perhaps this theory may say: The truth _is_ the credit. As a fact, a vast number of our human opinions--those, for instance, which relate to the past, or to the contents of other men's minds--appear, within the range of our personal experience, as credits whose value we, who believe the opinions, cannot hope ever to convert into the cash of experience. The banker who holds the bond not maturing within his own lifetime can, after all, if the bond is good, sell it to-day for cash. And that truth which he can personally and empirically test whenever he wants to test, is enough to warrant his act in accepting the credit. But I, who am confident of the truths of history, or of geology, or of physics, and who believe in the minds of other men,--I accept as valid countless opinions that are for me, in my private capacity and from an empirical point of view, nothing but irredeemable currency. In vain do I say: "I could convert these ideas into the cash of experience if I were some other man, or if I were living centuries ago instead of to-day." For the question simply recurs: In what sense are these propositions about my own possible experience true when I do not test their truth, --yes, true although I, personally, cannot test their truth? These credits, irredeemable in terms of the cash of my experience,--wherein consists their true credit value?

Here one apparently stands at the parting of the ways. One can answer this question by saying: "The truth of these assertions (or their falsity, if they are false) belongs to them whether I credit them or no, whether I verify them or not. Their truth or their falsity is their own character and is independent of my credit and my verification." But to say this appears to be, after all, just the intellectualism which so many of our modern pragmatists condemn. There remains, however, one other way. One can say: "The truth of the unverified assertions consists simply in the fact that, for our own private and individual ends, they are credited. Credit is relative to the creditor. If he finds that, on the whole, it meets his purpose to credit, he credits. And there is no truth, apart from present verifications except this truth of credit." In other words, that is true for me which I find myself accepting as my way of reacting to my situation.

This, I say, is a theory of truth which can be attempted. Consider what a magnificent freedom such a theory gives to all of us. Credit is relative to the creditor. To be sure, if ever the day of reckoning should come, one would be subject, at the moment of verification, to the constraints of experience. At such times, one would either get the cash or would not get it. But after all very few of our ideas about this great and wonderful world of ours ever are submitted to any such sharp tests. History and the minds of other men,

--well, our personal opinions about these remain credits that no individual amongst us can ever test for himself. As your world is mainly made up of such things, your view of your world remains, then, subject to your own needs. It ought to be thus subject. There is no absolute truth. There is only the truth that you need. Enter into the possession of your spiritual right. Borrow Nietzche's phraseology. Call the truth of ordinary intellectualism mere Sklavenwahrheit. It pretends to be absolute; but only the slaves believe in it. "Henceforth," so some Zarathustra of a new theory of truth may say, "I teach you Herrenwahrheit." Credit what you choose to credit. Truth is made for man, not man for truth. Let your life "boil over" into new truth as much as you find such effervescence convenient. When, apart from the constraints of present verification, and apart from mere convention, I say: "This opinion of mine is true." I mean simply: "To my mind, lord over its own needs, this assertion now appears expedient." Whenever my expediency changes, my truth will change.

But does anybody to-day hold just this theory of truth? I hesitate to make accusations which some of my nearest and dearest friends may repudiate as personally injurious. But this I can say: I find a great many recent theorists about truth talking in just this spirit so long as they feel free to glorify their spiritual liberty, to amuse their readers with clever assaults upon absolutism, and to arouse sympathy by insistence upon the human and the democratic attractiveness of the novel views of truth that they have to advance. Such individualism, such capriciousness, is in the air. Our modern theorists of truth frequently speak in this way. When their expressions of such views are criticized, they usually modify and perhaps withdraw them. What, as individuals, such teachers really mean, I have no right to say. Nobody but themselves can say; and some of them seem to say whatever they please. But this I know: Whoever identifies the truth of an assertion with his own individual interest in making that assertion may be left to bite the dust of his own confusion in his own way and time. The outcome of such essential waywardness is not something that you need try to determine through controversy. It is self-determined. For in case I say to you: "The sole ground for my assertions is this, that I please to make them,"--well, at once I am defining exactly the attitude which we all alike regard as the attitude of one who chooses not to tell the truth. And if, hereupon, I found a theory of truth upon generalizing such an assertion,--well, I am defining as truth-telling precisely that well-known practical attitude which is the contradictory of the truth-telling attitude. The contrast is not one between intellectualism and pragmatism. It is the contrast between two well-known attitudes of will,--the will that is loyal to truth as an universal ideal, and the will that is concerned with its own passing caprices. If I talk of truth, I refer to what the truth-loving sort of will seeks. If hereupon I define the true as that which the individual personally views as expedient in opinion or in assertion, I contradict myself, and may be left to my own confutation. For the position in which I put myself, by this individualistic theory of truth, is closely analogous to the position in which Epimenides the Cretan, the hero of the fallacy of the liar, was placed by his own so famous thesis.

VI.

And yet, despite all this, the modern assault upon mere intellectualism is well founded. The truth of our assertions is indeed definable only by taking account of the meaning of our own individual attitudes of will, and the truth, whatever else it is, is at least instrumental in helping us towards the goal of all human volition. The only question is whether the will really means to aim at doing something that has a final eternal meaning.

Herewith I suggest a theory of truth which we can understand only in case we follow the expressions of the third of the three modern motives to which I have referred. I have said that the new logic and the new methods of reasoning in the exact sciences are just now bringing us to a novel comprehension of our relation to absolute truth. I must attempt a very brief indication as to how this is indeed the case.

I have myself long since maintained that there is indeed a logic of the will, just as truly as there is a logic of the intellect. Personally, I go further still. I assert: all logic is the logic of the will. There is no pure intellect. Thought is a mode of action, a mode of action distinguished from other modes mainly by its internal clearness of self-consciousness, by its relatively free control of its own procedure, and by the universality, the impersonal fairness and obviousness of its aims and of its motives. An idea in the consciousness of a thinker is simply a present consciousness of some expression of purpose,--a plan of action. A judgment is an act of a reflective and self-conscious character, an act whereby one accepts or rejects an idea as a sufficient expression of the very purpose that is each time in question. Our whole objective world is meanwhile defined for each of us in terms of our ideas. General assertions about the meaning of our ideas are reflective acts whereby we acknowledge and accept certain ruling principles of action. And in respect of all these aspects of doctrine I find myself at one with recent voluntarism, whether the latter takes the form of instrumentalism, or insists upon some more individualistic theory of truth. But for my part, in spite, or in fact because of this my voluntarism, I cannot rest in any mere relativism. Individualism is right in saying, "I will to credit this or that opinion." But individualism is wrong in supposing that I can ever be content with my own will in as far as it is merely an individual will. The will to my mind is to all of us nothing but a thirst for complete and conscious self-possession, for fullness of life. And in terms of this its central motive, the will defines the truth that it endlessly seeks as a truth that possesses completeness, totality, self-possession, and therefore absoluteness. The fact that, in our human experience, we never meet with any truths such as completely satisfy our longing for insight, this fact we therefore inevitably interpret, not as any defect in the truth, but as a defect in our present state of knowledge, a limitation due to our present type of individuality. Hence we acknowledge a truth which transcends our individual life. Our concepts of the objectively real world, our ethical ideals of conduct, our estimates of what constitutes the genuine worth of life,--all these constructions of ours are therefore determined by the purpose to conform our selves to absolute standards. We will the eternal. We define the eternal. And this we do whenever we talk of what we call genuine facts or actualities, or of the historical content of human experience, or of the physical world that our sciences investigate. If we try to escape this inner necessity of our whole voluntary and self-conscious life, we simply contradict ourselves. We can define the truth even of relativism only by asserting that relativism is after all absolutely true. We can admit our ignorance of truth only by acknowledging the absoluteness of that truth of which we are ignorant. And all this is no caprice of ours. All this results from a certain necessary nature of our will which we can test as often as we please by means of the experiment of trying to get rid of the postulate of an absolute truth. We shall find that, however often we try this experiment, the denial that there is any absolute truth simply leads to its own denial, and reinstates what it denies.

The reference that I a little while since made to our assertions regarding the past, and regarding the minds of other men, has already suggested to us how stubbornly we all assert certain truths which, for every one of us, transcend empirical verification, but which we none the less regard as absolutely true. If I say: "There never was a past," I contradict myself, since I assume the past even in asserting that a past never was. As a fact our whole interpretation of our experience is determined, in a sense akin to that which Kant defined, by certain modes of our own activity, whose significance is transcendental, even while their whole application is empirical. These modes of our activity make all our empirical sciences logically possible. Meanwhile it need not surprise us to find that Kant's method of defining these modes of our activity was not adequate, and that a new logic is giving us, in this field, new light. The true nature of these necessary modes of our activity becomes most readily observable to us in case we rightly analyze the methods and concepts, not of our own empirical, but rather of our mathematical sciences. For in these sciences our will finds its freest expression. And yet for that very reason in these sciences the absoluteness of the truth which the will defines is most obvious. The new logic to which I refer is especially a study of the logic of mathematics.

VII.

That there are absolutely true propositions, the existence of the science of pure mathematics proves. It is indeed the case that, as Russell insists, the propositions of pure mathematics are (at least in general) hypothetical propositions. But the hypothetical character of the propositions of pure mathematics does not make the truth that a certain mathematically interesting consequent follows from a certain antecedent, in any way less than absolutely true. The assertion, "a implies b," where a and b are propositions, may be an absolutely true assertion; and, as a fact, the hypothetical assertions of pure mathematics possess this absolutely true character. Now it is precisely the nature and ground of this absoluteness of purely mathematical truth upon which recent research seem to me to have thrown a novel light. And the light which has appeared in this region seems to me to be destined to reflect itself anew upon all regions and types of truth, so that empirical and contingent, and historical and psychological and ethical truth, different as such other types of truth may be from mathematical truth, will nevertheless be better understood, in future, in the light of the newer researches into the logic of pure mathematics.

I can only indicate, in the most general way, the considerations which I here have in mind.

At the basis of every mathematical theory,--as, for instance, at the basis of pure geometry, or pure number theory,--one finds a set of fundamental concepts, the so-called "indefinables" of the theory in question, and a set of fundamental "propositions," the so-called "axioms" of this theory. Modern study of the logic of pure mathematics has set in a decidedly novel light the question: What is the rational source, and what is the logical basis of these primal concepts and of these primal propositions of mathematical theory? I have no time here to deal with the complications of the recent discussion of this question. But so much I can at once point out: there are certain concepts and certain propositions which possess the character of constituting the doctrine which may be called, in the modern sense, Pure Logic. Some of these concepts and propositions were long ago noted by Aristotle. But the Aristotelian logic actually took account of only a portion of the concepts of pure logic, and was able to give, of these concepts, only a very insufficient analysis. There is a similar inadequacy about the much later analysis of the presuppositions of logic which Kant attempted. The theory of the categories is in fact undergoing, at present, a very important process of reconstruction. And this process is possible just because we have at present discovered wholly new means of analyzing the concepts and propositions in question. I refer (as I may in passing state) to the means supplied by modern Symbolic Logic.

Well, the concepts of pure logic, when once defined, constitute an inexhaustible source for the constructions and theories of pure mathematics. A set of concepts and of propositions such as can be made the basis of a mathematical theory is a set possessing a genuine and unquestionable significance if, and only if, these concepts and these propositions can be brought into a certain definite relation with the concepts and propositions of pure logic. This relation may be expressed by saying that if the conditions of general logical theory are such as to imply the valid possibility of the mathematical definitions and constructions in question, then--but only then--are the corresponding mathematical theories at once absolutely valid and significant. In brief, pure mathematics consists of constructions and theories based wholly upon the conceptions and propositions of pure logic.

The question as to the absoluteness of mathematical truth hereupon reduces itself to the question as to the absoluteness of the truths of pure logic.

Wherein, however, consists this truth of pure logic? I answer, at once, in my own way. Pure logic is the theory of the mere form of thinking. But what is thinking? Thinking, I repeat, is simply our activity of willing precisely in so far as we are clearly conscious of what we do and why we do it. And thinking is found by us to possess an absolute form precisely in so far as we find that there are certain aspects of our activity which sustain themselves even in and through the very effort to inhibit them. One who says: "I do not admit that for me there is any difference between saying yes and saying no,"--says "no," and distinguishes negation from affirmation, even in the very act of denying this distinction. Well, affirmation and negation are such self-sustaining forms of our will activity and of our thought activity. And such self-sustaining forms of activity determine absolute truths. For instance, it is an absolute truth that there is a determinate difference between the assertion and the denial of a given proposition, and between the doing and the not doing of a given deed. Such absolute truths may appear trivial enough. Modern logical theory is for the first time making clear to us how endlessly wealthy in consequences such seemingly trivial assertions are.

The absoluteness of the truths of pure logic is shown through the fact that you can test these logical truths in this reflective way. They are truths such that to deny them is simply to reassert them under a new form. I fully agree, for my own part, that absolute truths are known to us only in such cases as those which can be tested in this way. I contend only that recent logical analysis has given to us a wholly new insight as to the fruitfulness of such truths.

VIII.

An ancient example of a use of that way of testing the absoluteness of truth which is here in question is furnished by a famous proof which Euclid gave of the theorem, according to which there exists no last prime number in the ordinal sequence of the whole numbers. Euclid, namely, proved this theorem by what I suppose to be one device whereby individual instances of absolute truths are accessible to us men. He proved the theorem by showing that the denial of the theorem implies the truth of the theorem. That is, if I suppose that there is a last prime number, I even thereby provide myself with the means of constructing a prime number, which comes later in the series of whole numbers than the supposed "last" prime, and which certainly exists just as truly as the whole numbers themselves exist. Here, then, is one classic instance of an absolute truth.

To be sure Euclid's theorem about the prime numbers is a hypothetical proposition. It depends upon certain concepts and propositions about the whole numbers. But the equally absolute truth that the whole numbers themselves form an endless series, with no last term, has been subjected, in recent times, to wholly new forms of reëxamination by Dedekind, by Frege, and by Russell. The various methods used by these different writers involve substantially the same sort of consideration as that which Euclid already applied to the prime numbers. There are certain truths which you cannot deny without denying the truth of the first principles of pure logic. But to deny these latter principles is to reassert them under some other and equivalent form. Such is the common principle at the basis of the recent reëxamination of the concept of the whole numbers. Dedekind, in showing that the existence of the dense ordinal series of the rational numbers implies the existence of the Dedekind Schnitte of this series, discovered still another absolute, although of course hypothetical, truth which itself implies the truth of the whole theory of the so-called real numbers. Now all such discoveries are indeed revelations of absolute truth in precisely this sense, that at the basis of all the concepts and propositions about number there are concepts and propositions belonging to pure logic; while if you deny these propositions of pure logic, you imply, by this very denial, the reassertion of what you deny. To discover this fact, to see that the denial of a given proposition implies the reassertion of that proposition, is not, as Kant supposed, something that you can accomplish, if at all, then only by a process of mere "analysis." On the contrary, Euclid's proof as to the prime numbers, and the modern exact proofs of the fundamental theorems of mathematics, involve, in general, a very difficult synthetic process,--a construction which is by no means at first easy to follow. And the same highly synthetic constructions run through the whole of modern logic.

Now once again what does one discover when he finds out such absolute truths? I do not believe, as Russell believes, that one in such cases discovers truths which are simply and wholly independent of our constructive processes. On the contrary, what one discovers is distinctly what I must call a voluntaristic truth,--a truth about the creative will that thinks the truth. One discovers, namely, that our constructive processes, viewed just as activities, possess a certain absolute nature and conform to their own self-determined but, for that very reason, absolute laws. One finds out in such cases what one must still, with absolute necessity, do under the presupposition that one is no longer bound by the constraints of ordinary experience, but is free, as one is in pure mathematics free, to construct whatever one can construct. The more in such cases, one deals with what indeed appear to be, in one aspect, "freie Schöpfungen des menschlichen Geistes," the more one discovers that their laws, which are the fundamental and immanent laws of the will itself, are absolute. For one finds what it is that one must construct even if one denies that, in the ideal world of free construction which one is seeking to define, that construction has a place. In brief, all such researches illustrate the fact that while the truth which we acknowledge is indeed relative to the will which acknowledges that truth, still what one may call the pure form of willing is an absolute form, a form which sustains itself in the very effort to violate its own laws. We thus find out absolute truth, but it is absolute truth about the nature of the creative will in terms of which we conceive all truths.

Now it is perfectly true that such absolute truth is not accessible to us in the empirical world, in so far as we deal with individual phenomena. But it is also true that we all of us conceive the unity of the world of experience--the meaning, the sense, the connection of its facts--in terms of those categories which express precisely this very form of our creative activity. Hence, although every empirical truth is relative, all relative truth is inevitably defined by us as subject to conditions which themselves are absolute. This, which Kant long ago maintained, gets a very new meaning in the light of recent logic,--a far deeper meaning, I think, than Kant could conceive.

In any case, the new logic, and the new mathematics, are making us acquainted with absolute truth, and are giving to our knowledge of this truth a clearness never before accessible to human thinking. And yet the new logic is doing all this in a way that to my mind is in no wise a justification of the intellectualism which the modern instrumentalists condemn. For what we hereby learn is that all truth is indeed relative to the expression of our will, but that the will inevitably determines for itself forms of activity which are objectively valid and absolute, just because to attempt to inhibit these forms is once more to act, and is to act in accordance with them. These forms are the categories both of our thought and of our action. We recognize them equally whether we consider, as in ethics, the nature of reasonable conduct, or, as in logic, the forms of conceptual construction, or, as in mathematics, the ideal types of objects that we can define by constructing, as freely as possible, in conformity with these forms. When we turn back to the world of experience, we inevitably conceive the objects of experience in terms of our categories. Hence the unity and the transindividual character which rightly we assign to the objects of experience. What we know about these objects is always relative to our human needs and activities. But all of this relative knowledge is--however provisionally--defined in terms of absolute principles. And that is why the scientific spirit and the scientific conscience are indeed the expression of motives, which you can never reduce to mere instrumentalism, and can never express in terms of any individualism. And that is why, wherever two or three are gathered together in any serious moral or scientific enterprise, they believe in a truth which is far more than the mere working of any man's ephemeral assertions.

In sum, an absolute truth is one whose denial implies the reassertion of that same truth. To us men, such truths are accessible only in the realm of our knowledge of the forms that predetermine all of our concrete activities. Such knowledge we can obtain regarding the categories of pure logic and also regarding constructions of pure mathematics. In dealing, on the other hand, with the concrete objects of experience, we are what the instrumentalists suppose us to be, namely, seekers for a successful control over this experience. And as the voluntarists also correctly emphasize, in all our empirical constructions, scientific and practical, we express our own individual wills and seek such success as we can get. But there remains the fact that in all these constructions we are expressing a will which, as logic and pure mathematics teach us, has an universal absolute nature,--the same in all of us. And it is for the sake of winning some adequate expression of this our absolute nature, that we are constantly striving in our empirical world for a success which we never can obtain at any instant, and can never adequately define in any merely relative terms. The result appears in our ethical search for absolute standards, and in our metaphysical thirst for an absolute interpretation of the universe,--a thirst as unquenchable as the over-individual will that expresses itself through all our individual activities is itself world-wide, active, and in its essence absolute.

In recognizing that all truth is relative to the will, the three motives of the modern theories of truth are at one. To my mind they, therefore, need not remain opposed motives. Let us observe their deeper harmony, and bring them into synthesis. And then what I have called the trivialities of mere instrumentalism will appear as what they are,--fragmentary hints, and transient expressions, of that will whose life is universal, whose form is absolute, and whose laws are at once those of logic, of ethics, of the unity of experience, and of whatever gives sense to life.

Tennyson, in a well-known passage of his "In Memoriam," cries:

> "Oh living Will that shalt endure
>   When all that seems shall suffer shock,
>   Rise in the spiritual rock,
> Flow through our deeds and make them pure."

That cry of the poet was an expression of moral and religious sentiment and aspiration; but he might have said essentially the same thing if he had chosen the form of praying: Make our deeds logical. Give our thoughts sense and unity. Give our Instrumentalism some serious unity of eternal purpose. Make our Pragmatism more than the mere passing froth of waves that break upon the beach of triviality. In any case, the poet's cry is an expression of that Absolute Pragmatism, of that Voluntarism, which recognizes all truth as the essentially eternal creation of the Will. What the poet utters is that form of Idealism which seems to me to be indicated as the common outcome of all the three motives that underlie the modern theory of truth.

Chapter IV

ERROR AND TRUTH

Editor's Note: See the editor's note under *Axiom* below, p. 125 .

I. Introduction.

Both in its philosophical and in its popular acceptation the word 'error' is applied to false opinions. But the popular usage also gives to the term a still wider meaning, whereby it includes not only false opinions, but numerous forms of practical failure, and of defective conduct, whose relations to conscious beliefs are by no means constant or easily discoverable. The derivation of the word illustrates the naturalness of associating the conception of false opinion with the idea of some such act as wandering, or straying, or missing the way. It seems, therefore, as if a first approach to a sharper definition of 'error' would be aided by clearly distinguishing between the practical and the theoretical applications, and then confining the philosophical use of the term, so far as possible, to theoretical errors. But we shall find it impossible to define even theoretical error without reference to some genuinely practical considerations. However much we try to avoid popular confusions, we shall be led in the end to a concept of error which can be stated only in teleological terms, and which involves the idea of action for an end, and of a certain defect in the carrying out of such action.

The present, article, after distinguishing, as far as possible, the concept of theoretical errors, or of false opinions, from the popular concept of practical errors, and after stating some of the best known views regarding what a false opinion is, will seek to indicate the nature of a solution of the problem in terms of a doctrine about the relation of the cognitive to the volitional processes.

2. Practical errors and false opinions.--When one emphasizes the practical aspect of an error, one sometimes makes use of the more drastic word 'blunder.' A blunder is something which involves serious maladjustment, defect in conduct. Errors in the sense of blunders may be due to false opinions, or may even very largely consist of such. On the other hand, they need not involve false opinions, and must involve actions which do not attain their goal. These actions may be only partly voluntary; but the relation of their defective aspects to the accompanying voluntary processes is what makes us call them errors. Thus, we speak of the error or blunder of the marksman who misses his mark; of the player who fails to score, or who permits his opponent to score when the game calls for some device for hindering the opponent from scoring. We speak of the musician's error when he sings or plays a false note. Such errors may, but often do not, result from, or accompany, false opinions or misjudgments. Thus one may fail as marksman, as player, or as musician, either through misjudgments or through defects of physical training, of temporary condition, of mood, or of attention--defects which may involve no false opinions whatsoever.

In the moral realm, the relations between such practical errors on the one hand and false opinions on the other are especially momentous and intricate. Here, in fact, the theory of moral error involves all the main problems about the relations between knowledge and action. A sin is very generally called an error. 'We have erred and strayed from thy ways like lost sheep.' The error is, first of all, practical. It has also some relation to knowledge. Yet, since sin appears to depend upon some degree of knowledge of right, the 'error' in question does not merely result from a false opinion about what one's duty is. On the other hand, that sin involves 'unwisdom,' and so does in some respect depend upon false opinions, is very generally asserted. Any careful discussion of those practical errors which have a moral significance will, therefore, show that it is no merely accidental confusion which has led us to our use of a word derived from our experience of wanderings from the right path as a term which is also to be applied to false opinions. Opinions certainly express themselves in actions; and voluntary actions are guided by opinions. The resulting relations of cognition and volition, especially in the moral world, are amongst the most complex and intimate which are known to us anywhere. They are relations which we can neither ignore nor wholly disentangle. Hence the clear separation of theoretical error and practical error, at least in the moral world, is impossible. For sin involves both theoretical and practical defects.

We can, however, make some approach to such a separation of the theoretical and practical aspects of error if we turn for aid to a very different realm, namely, formal logic. The distinction between true and false propositions involves certain well-known general relations, such as formal logic considers and analyzes. We may use these relations for what they are worth in attempting to define what a false opinion is. Having thus laid a basis for further analysis, we may attempt to clear the way through some of the more complex regions of the problem of error.

The distinction between true opinion and false opinion obviously depends upon, but also is obviously not identical with, the formal logical distinction between true and false propositions. This close relation and important difference between these two distinctions appear upon a brief study of the considerations which formal logic employs in dealing with the concepts of truth and falsity. True and false are, for the formal logician, predicates belonging to propositions, quite apart from any question as to whether anybody believes or asserts those propositions. With regard to the predicates 'true' and 'false,' formal logic uses, upon occasion, the following well-known principles, which we may here provisionally accept as a basis for further inquiry: (1) every proposition (supposing its meaning to be precise) is either true or false, and cannot be both true and false; (2) to every proposition there corresponds a determinate proposition which is the contradictory of the first proposition; (3) the relation of contradictories is reciprocal or 'symmetrical'; (4) of two contradictory propositions, one is true and the other is false. These may be here regarded, if one chooses, merely as defining principles, explaining what one means by propositions, and how one proposes to use the logical predicates 'true' and 'false'.

Granting these purely formal principles, of which all exact reasoning processes make constant use, it is obvious that propositions taken collectively as a system constitute an ideal realm wherein to every truth there uniquely corresponds its contradictory falsity, and to every false proposition its contradictory true proposition. The realms of truth and falsity are thus formally inseparable. To know that a given proposition is false is to know that the corresponding contradictory is true, and vice versa. Omniscience regarding the realm of truth would, therefore, equally involve knowing true propositions as true and false propositions as false; nor could the one sort of knowledge be defined or real without the other.

But no such formal logical necessity appears to connect true opinion and error. No one can know that $2 + 2 = 4$ is true without thereby knowing that $2 + 2 \neq 4$ (that is, the contradictory of the former assertion) is false. But we can conceive of a computer who should never make any errors in computation; and such a computer might even be supposed so perfect, in the possession of some superhuman infallibility of computation, as not even to know what it would be to err in his additions. We ourselves, when we use the assertion $2 + 2 = 4$ as an example of a peculiarly obvious proposition of computation, find this bit of summation one about which it is rare or difficult for a man 'in his sober senses' to err. Yet for us the knowledge of the truth of the proposition 2 + 2 = 4 is logically inseparable from the knowledge of the falsity of the contradictory of this proposition.

In sum, then, true and false propositions are logically inseparable. To possess a knowledge of truth is, therefore, inseparable from the possession of a knowledge of what falsity is, and of what false propositions mean. But a being can be supposed to know truth and falsity, and their distinctions and relations, without having any tendency to fall a prey to error. At all events, no purely formal logical reasons, such as for the moment concern us, can be given for supposing that a being who is capable of knowing truth should be capable of falling into error. The more concrete distinction between true opinion and error must, therefore, be different from the formal logical difference between truth and falsity. The latter may be viewed as a logically necessary distinction between inseparable objects. The former must be due to motives or causes, and must imply mental tendencies and situations of which formal logic, taken in its deliberate abstraction from the fullness of life, gives no account.

The concept of a false opinion is thus obviously distinct from that of a false proposition, and not every true opinion requires that the corresponding false opinion should be held by somebody. It is the purpose of advancing science, of education, of the propagation of truth, to diminish and, so far as may be, 'to banish error' from the minds of men. If this purpose were somehow miraculously attained, there would be as many false propositions in the formal logician's ideal realm of truth and falsity as there ever were; but human errors would have ceased.

3. The leading definitions of error.--To define false opinion, hereupon, as the acceptance or the mistaking of false propositions for true ones, or of true for false ones, is a familiar device of philosophers, but it throws no light upon the real nature of error. For,

to mistake a falsity for a truth, to accept a false opinion as true--what is this but simply to make a mistake, or to hold a false opinion? This supposed definition is but a tautology. Not thus is the nature of error to be clarified. Further light upon the subject can be obtained only through (1) defining more exactly the distinction between true and false propositions; and (2) showing upon what further distinctions the conception of error depends. Some of the best known efforts to accomplish this result must next be summarily stated and criticized.

(i.) The 'Correspondence theory of truth and falsity' and the definitions of error based upon it deserve to be stated, because they are familiar, and because they have formed the starting-point for supplementary doctrines and definitions and for corrections. According to the view now in question, a proposition is true if it reports, or describes, or portrays 'facts as they are.' The emphasis is laid upon the 'as.' A true idea 'corresponds' in its structure to the thing, or reality, or fact of which it is a true idea; a true proposition is one which asserts that an idea does thus correspond to the facts, when it actually so corresponds. Or, again, if the account given by a proposition conforms to the structure of the facts of which it attempts to furnish an ideal portrayal, the proposition is true. Thus, a proposition may relate to the number in a real flock of sheep. In this case an idea, gained by counting the sheep, is first formed, and then the assertion is made that this numerical idea represents the real number of sheep present in the flock. The correspondence of the idea with the facts constitutes that to which the assertion is committed. If the correspondence exists, the assertion is true.

Such being (according to the 'correspondence' theory) the nature of truth, error takes place when, because of inadequate observation of the sheep, or because of some other psychological defect on the part of the one who counts, a numerical idea which does not correspond to the real number of the sheep arises in the mind that is subject to the error; while, because of these or of still other psychological motives, the false proposition 'Such is the number of sheep,' comes to be asserted. That the correspondence does not exist makes the proposition false. That this non-existent correspondence is asserted and believed to exist constitutes the essence of the error.

In order to understand what error is, and how it arises, one therefore needs, according to this view, to analyze the nature of belief, and the motives which lead the erring mind to make assertions. From this point onwards, the definition and the theory of error have always required the consideration of various associative, affective, or volitional factors of the process of making and believing assertions--factors of which pure logic considered in its usual abstraction, can give no account. In brief, the nature of truth and falsity once having been thus defined, the nature of error depends upon some disposition to accept or to assert an untrue proposition --a disposition which cannot be due to the merely logical nature of the untruth itself, but must be referred to the prejudices, the feelings, the ignorance, the wilfulness, or the other psychological fortunes of the erring subject.

What further accounts, upon this basis, have been attempted as explanations of the essence of error, there is here no space to set forth at length. A few points must be noted. One may assert: (1) that error in such a case as the foregoing, or in the more complex cases of superstitions, supposed theological heresies, false philosophies, errors in scientific opinion, false political doctrines, etc., may be mainly due to a negative cause--the mere ignorance of the erring subject, his lack of 'adequate ideas,' the absence of correct and sufficient portrayals of fact. What a man lacks he cannot use. If he has no ideas that correspond with the facts in question, how can he make true assertions? Error is then, at least in the main (according to the view now in question), due to privation. For instance, I may not even attempt to count the sheep in the flock. I may merely guess at random. In such a case, error seems to be due merely, or mainly, to my lack of ideas. Such a negative theory of error was worked out by Spinoza, and applied by him, as far as possible, to decidedly complex cases. Naturally, according to Spinoza, 'the order and connexion of ideas' corresponds to 'the order and connexion of things.' This, for Spinoza, is the case with even the most worthless of our human imaginations. But for psycho-physical reasons, which Spinoza discusses at length, most ideas of the ordinary man, relating to his world, are extremely 'inadequate'; that is, such ideas correspond only to very fragmentary aspects of the real world. The majority of men live 'ignorant of God and of themselves and of things.' This ignorance prevents them from possessing the stock of ideas which could furnish the basis for true opinions. Men fill the void with errors. Yet none even of their errors is without basis in fact. They simply judge, without restraint, concerning that of which they know not, just because they know so little. This doctrine of error as ignorance, if accepted, would give us the most purely and completely theoretical definition of error which has ever been offered.

Plainly, however, ignorance is not of itself error. I cannot err concerning facts of which I know so little as to have no idea whatever about them; just as I cannot, in a speech, make grammatical blunders of whose existence I have never heard. Some other factor than ignorance determines the actual acceptance and utterance of false propositions. This even Spinoza himself has in the end to recognize. In his study of the errors of human passion, he makes the mechanical associative process, and the resulting passions themselves, factors in the genesis of error. Thus we are inevitably led to further theories.

One may assert: (2) that error is due to whatever moves the will of the erring subject to make the assertions even in the absence of ideas that correspond to real objects. This volitional theory of error played a considerable part in Scholastic doctrine; was obviously useful in giving reasons for the moral condemnation of the errors of heretics, infidels, and schismatics; and has, in fact, an obvious and important basis in the psychology of opinion. Descartes recognized it in connexion with his own form of the doctrine of the freedom of the will. Spinoza, who rejected the theory of free will, and defined both intellect and will in terms of his psycho-physical theory of the associative process, still on occasion was obliged, as just pointed out, to use his own version of the doctrine of 'human bondage' as an explanation of the fatal errors into which the play of our inadequate ideas and of our passions leads us. In other forms this theory of error is widely accepted. From this point of view an error is a wilful assertion of a false proposition--an assertion made possible, indeed, by the erring subject's ignorance of the ideas that do correspond with reality, but positively determined by his willingness to assert. False beliefs are thus due to a combination of ignorance with the will to believe.

One may insist (3) that the affective processes which condition the mood called 'belief' are the principal factors in making a false proposition, when it chances to be suggested, seem plausible. Where error is propagated by social contagion, or is accepted through reverence for authority, not so much the will as the emotional life of the erring subject seems to be the factor which makes error possible. Here, according to the previous view, ignorance of ideas that do correspond with reality is a condition of error, but constitutes neither its essence nor its sufficient cause. An error, according to the present view, is a false opinion which, because of its appeal to the sentiments, the feelings, the prejudices, of the erring subject, because it is harmonious with his social interests or with his private concerns, wins the subject over to the state of mind called belief.

One may further maintain: (4) that the principal cause of error is whatever associative, perceptual, or imaginative process gives such liveliness, strength, and persistence to ideas which as a fact do not correspond with reality, that the erring subject is forced, in the absence of sufficient corrective ideas, or (to use Taine's expression) for lack of 'reductors,' to regard these ideas as representatives of reality. Theories of error founded upon this view have played no small part in the psychiatrical literature which deals with the genesis of pathological forms of error, and have been prominent in the teachings of the Associationist school generally. From this point of view, an error is a false proposition whose assertion is forced upon the erring subject through the mechanism of association, and mainly because no other ideas than those which this assertion declares to correspond with the facts can win a place in the subject's mind when he thinks of the topic in question.

The foregoing accounts of the nature and source of error have all been stated with explicit reference to the 'correspondence' theory of truth. This theory supposes that the test of truth is the actual conformity of a representative idea with the object which it is required to portray. Idea and object are viewed as distinct and separable facts, just as a man and his portrait or photograph are possessed of a separate existence. The representative idea is external to the object. Truth depends upon a certain agreement between such mutually external facts. And, just as the idea to whose truthfulness as a representation a proposition is committed is external to its object, so, as we have now seen, the motives which lead us to error appear, in the accounts thus far given, to be external to the meaning, and to the truth or falsity, of ideas and propositions. The falsity of a proposition, so far as we have yet seen, gives no reason why the error involved in believing that proposition should be committed. The truth of a proposition, also, in no wise explains why the true proposition comes to be believed--unless, indeed, with Spinoza, one comes to accept, for metaphysical reasons, a theory that ideas are by nature in agreement with objects. In case, however, one does accept the latter theory without any limitation, then error can be defined only in negative terms as due to mere absence of ideas. Such an account of error, as we have also seen, is incapable of telling us what it is, and is inadequate to explain the most familiar facts about its occurrence.

If, then, the truth or the falsity of a proposition does not of itself explain why we come to get a true or false belief, the existence of error, for one who accepts the correspondence theory of truth has to be explained by psychological motives which are as external to the logical meanings of true and false propositions as the ideas of the correspondence theory of truth are external to their objects. Some propositions are true. Their contradictories are false. So far, we have a system of facts and relations that seems according to this account, to be wholly independent of the psychological processes of anybody. But of these true and false propositions, some are believed by men. If the propositions believed are true, we have not explicitly considered in the foregoing the psychology of the process by which they come to be believed. But, if the beliefs are beliefs in false propositions, some accounts of how the errors arise have been suggested. These accounts all appeal to motives which do not result from the falsity of the propositions, but from the feelings, the will, or the associative processes of the erring subject--all of them influences which are due not to the logical distinctions between true and false, but to the mental fortunes of the believer.

Unfortunately, however, since the true beliefs of the subject must also have their psychology, quite as much as the false beliefs, and since the will, the feelings, the associative processes, the conditions which determine 'lively ideas,' and the like, must be equally effective when true propositions are believed as when false beliefs triumph--all the foregoing accounts leave us dissatisfied should we be led to ask: What are the processes which prevent error and give us true beliefs? For, apart from Spinoza's assumption of the universal agreement between 'the order and connexion of ideas and the order and connexion of things'--an assumption which makes error in any but a purely negative sense impossible--the truth of a proposition is a fact which in no wise explains why we mortals should come to believe that proposition to be true. And, if we explain the true belief as due to the will, the feelings, the associative or other psychological processes of the subject, these factors, as the theories of error so far stated have insisted, work as well to produce error as to beget true opinion. The one thing of which we have so far given no account is the way in which the difference between true opinion and error arises--the factor which is decisive in determining whether a given state of opinion, in a given subject, shall be one which accepts true propositions or, on the contrary, embraces error.

Of course, the need of such an account has frequently been felt by the partisans of the 'correspondence' theory of truth. Innumerable portrayals exist of the ways in which conformity of idea and object can be furthered or attained by psychological processes. Ideas can be made 'clear and distinct,' observations of the object can be rendered careful, prejudice can be kept in abeyance, feeling can be controlled, judgment can be suspended until the evidence is incontestable, and so on. By such means error can be more or less completely avoided, and agreement with the object can be progressively obtained. There is no doubt of the practical importance of such advice. There is also no doubt that the processes of control and of clarification which are in question are psychological processes, which the inquiring subject can find or produce within himself. It becomes plain, however, as one reflects, that to insist upon such matters is more or less to modify, and in the end to abandon, the representative theory of truth as consisting merely in the conformity of ideas to objects that are external to these ideas.

For how does one know, or why does one judge, that clear ideas, careful observations, the avoidance of prejudice, the suspension of judgment, and the other psychological devices of the truth-seeker, actually tend to make the subject escape from error, and win true opinion? Is it because, from some point of view external both to the object and to the ideas of the subject, one observes how the subject gradually wins a closer conformity with his object through using the better devices, and through avoiding the mental sources of error? If so, then whoever has this point of view, external both to the object and to the cognitive process, is already somehow acquainted with the constitution of the object, and is aware what propositions are true about the object quite apart from the psychological fortunes of the poor subject, whose escape from error is to be aided by such wise counsels. As a fact, philosophers who give such counsels very often behave for the moment as if they, at least, had not to wait for a slowly acquired conformity with the nature of reality, but were already assured of their own grasp of the object, and were therefore able to give such good advice to the erring psychological subject. No purely psychological theory of the way in which a conformity to an external object can be gradually acquired through clear ideas, freedom from prejudices, and so on, can serve to explain how the critic of human truth and error has himself acquired his assumed power to see things as they are, and thus to guide the psychological subject in the right path. That sort of attainment of truth which this theory attributes to the philosopher who teaches it is just what it does not explain.

In fact, a little reflexion shows that, when we hold, as we very rightly do, that a certain wise conduct of our ideas, feelings, will, observations, processes of recording observations, and other such mental enterprises helps us towards truth, and aids us to avoid error, we are comparing, not ideas with merely external objects, so much as less coherent with more coherent, unified, clear, and far-reaching forms of experience, of cognition--in general, of insight. If we once see this fact, we have to alter our definition of truth, and herewith our definitions both of true opinion and of error.

Truth cannot mean mere conformity of idea to external object; first, because nobody can judge an idea merely by asking whether it agrees with this or with that indifferent fact, but only by asking whether it agrees with that with which the knowing subject meant or intended it to agree; secondly, because nobody can look down, as from without, upon a world of wholly external objects on the one hand, and of his ideas upon the other, and estimate, as an indifferent spectator, their agreement; and thirdly, because the cognitive process, as itself a part of life, is essentially an effort to give to life unity, self-possession, insight into its own affairs, control of its own enterprises--in a word, wholeness. Cognition does not intend merely to represent its object, but to attain, to possess, and to come into a living unity with it.

Accordingly, the theories of error which have been founded upon the 'correspondence' theory of truth must be, not simply abandoned, but modified, in the light of a richer theory of truth. A true proposition does, indeed, express a correspondence between idea and object, but it expresses much more than this.

(ii.) Another definition of truth, which has its foundation far back in the history of thought, but which has been of late revised and popularized under the names of Pragmatism, Humanism, and Instrumentalism, may next be mentioned.

According to this view, an idea is essentially something that tends to guide or to plan a mode of action. A proposition expresses the acceptance of such a mode of action, as suited to some more or less sharply defined end. Now, a mode of action inevitably leads to consequences, which arise in the experience of the active subject. These consequences may be called the 'workings' of that idea which tended to guide or to plan this mode of action. These workings may agree or disagree with the intent of the idea. If the idea agrees with its expected workings, that idea is true, and with it the proposition which accepts that idea as suited to its own ends is true; otherwise the idea and the proposition are erroneous. Such is the definition of truth which is characteristic of Pragmatism.

The case of the right or wrong counting of the flock of sheep will serve to illustrate the present theory of true opinion and of error quite as readily as to exemplify the representative theory of the same matters. A flock of sheep is not merely an external object to be portrayed. It is, to the one who counts it, an interesting object of human experience. He counts it in order to be ready to estimate his possessions, to sell or to buy the flock, to know whether he needs to hunt for lost sheep, or because of some other concrete purpose. His counting gives him an idea, perhaps of what he ought to ask of a purchaser, or of a plan for the shearing or for the market, perhaps of whether he ought to search for missing sheep. When he accepts and asserts that some determinate number represents the actual number of the flock, he, no doubt, takes interest in the correspondence between the idea and the object; yet his real object is not the indifferent external fact, but the flock of sheep as related to his own plans of action and to the practical results of these plans. The only test of the truth of his count, and, in fact, the only test that, when he counts, he proposes to accept, is that furnished by the workings of his count. Does his idea of the number of the sheep, when accepted, lead to the expected results? One of these results, in many cases, is the agreement of his own count with that made by somebody else, with whom he wishes to agree concerning a sale or some other enterprise. Or, again, he expects the enumeration which he makes at one time to agree with the result obtained at some other time when he counts the flocks anew. Furthermore, a habit of inaccurate counting betrays itself, in the long run of business, in the form of failure to get expected profits, or in the form of a loss of sheep whose straying is at one time not noticed because of inaccurate counting; while later experience shows, in the form of the experience which traces the loss, the non-correspondence of expectations and results. Such expectations, tests, and agreements define the sort of truth that is sought.

What so simple and commonplace an instance illustrates, the whole work of the natural sciences, according to the pragmatist, everywhere exemplifies. The Newtonian theory of gravitation is accepted as true because its ideas lead, through computations, to workings which agree with observation. The corpuscular theory of light was rejected because certain of its consequences did not agree with experience. The same process of testing hypotheses by a comparison of expectations with outcome can be traced throughout the entire range of empirical investigation.

As to the cause and essence of error, upon the basis of this theory of truth, there can be, according to the pragmatist, no very subtle difficulties to solve. The whole matter is, upon one side, empirical; upon the other side, practical. Experience runs its course, however it does. We, the truthseekers, are endeavouring to adjust our actions to empirical happenings by adapting our expectations, through the definition of our ideas, and through the forming and testing of our hypotheses, to the observed facts as they come. As we are always in our practical life looking to the future, and are seeking the guidance which we need for our undertakings, our propositions are hypotheses to the effect that certain ideas will, if tested, agree with certain expected workings. If the test shows that we succeed, then, just when and in so far as we succeed, our propositions prove to be then and there true. If we fail, they prove to be errors. Truth and falsity, and, consequently, true opinion and error, are not 'static' properties or fixed classifications of our ideas or of our hypotheses. Both the ideas and the propositions 'come true' or 'fail to come true' through the fluent and dynamic process of the empirical test. Thus every truth is true and every falsity false, relatively to the time when, and the purpose for which, the individual idea or hypothesis is tested.

Absolute truth or permanent truth, and equally absolute falsity or permanent falsity, are from this point of view, purely abstract and ideal predicates, useful sometimes for formal purposes when we choose to define our purposes in terms of logical or of mathematical definitions. 'Concrete' truth and error are of the nature of events, or series of events, or of 'the long run' of experience. That many of our ideas should not 'work,' or that many of our hypotheses should result in disappointed expectation is, for the pragmatist, merely an empirical fact, requiring a special explanation no more than do the marksman's missing or the player's failure to score. We are not perfectly skillful beings; experience is often too fluent or too novel for our expectations. The wonder is rather that this is not more frequently the case. That man is as skillful a player as he is of the game of ideal expectations and anticipated consequences is a matter for congratulation. But failure is as natural an event as is success.

The traditional accounts of the psychology of error mentioned above are readily accepted by Pragmatism, precisely in so far as they are indeed accounts which experience justifies. No doubt, ignorance is a source of error. We are, in fact, ignorant of all except what experience, in one way or another, permits us upon occasion to prove by actual trial. This ignorance permits errors, in the form of false expectations, to arise. Prejudice, emotion, wilfullness, and the associative process unite to engender expectations which may prove to be false. Nor is there any known cause that uniformly assures the attainment of truth. The difference between success and failure in our adjustment to our situation is simply an empirical difference. We have to accept it as such. No deeper account can be given than experience warrants.

The result of the pragmatist's definition of error obviously forbids any sharp distinction between theoretical and practical errors. The presence or absence of conscious ideas, of definite expectations, of articulate hypotheses, remains (in case of our always more or less practically significant maladjustments of our acts to our situation) as the sole criterion for distinguishing between erroneous opinions, on the one hand, and blunders that are made, on the other hand--merely as the fumbling player may fail to get the ball, or as the nervous musician may strike the false note--blindly, and without knowing why one fails, or what false idea, if any, guided one to the failure. This reduction of all errors to the type of practical maladjustments is a characteristic feature of Pragmatism.

If the 'correspondence' theory of truth makes the distinction between true and false opinions something that is quite external to the logical distinction between true and false propositions, the pragmatist's theory of truth and error in propositions seems, on the contrary, to go as far as is possible to annul altogether the difference between these two sorts of distinctions. For the pragmatist it is merely a formal device of the logician to regard truth and error as in any sense permanent properties, or predicates of the supposed entities called propositions. What actually occurs, what empirically happens, is a series of concrete agreements and disagreements between expectations and results. These happenings or 'the long run' of such happenings, constitute all that is concretely meant by truth and error. Whether one says, 'This proposition is true or false' or 'This opinion is true or false,' the concrete fact to which one refers is the sequence of testings to which ideas are submitted when their expected workings are compared with the expectations. Since logicians like to abstract certain 'forms' from the matter of life, they may, if they choose, define the entities called true and false propositions, and then leave to the students of the concrete the study of the fortunes of mere opinions. As a fact, however, according to Pragmatism, propositions live only as opinions in process of being tested. The distinctions with which we began this discussion have their own provisional usefulness, but only as abstractions that help to prepare the way for understanding life.

A proposition becomes true in the concrete when the opinion that is true leads to expected workings, and becomes false when the belief in it leads to workings which do not agree with expectations.

Such, in sum, is the pragmatist's solution of our initial problem. It emphasizes very notable facts regarding the relations between logic and life, and between thought and volition. Yet it fails to satisfy. For it can only be stated by constantly presupposing certain assertions about experience, about the order, the inter-relations, the significance, and the unity of empirical facts to be true, although their truth is never tested, in the pragmatist's sense of an empirical test, at any moment of our experience.

Thus, it has been necessary to assume, even in stating the view of Pragmatism about truth, that ideas can be formed at one time, and submitted to the test of experience at another time, and perhaps by another person, just as Newton's hypotheses were formed by him, but were tested, not only by himself, through a long course of years, but by later generations of observers. It has been necessary to assume that one can form expectations today, and compare them with facts to-morrow, or next year, or after whatever length of time the conditions make possible. But this assumption requires the truth of the proposition that the meaning, the object, the purpose, the definition of the ideas, and expectations of one moment or period of time, or person, not only can be but are identical with the meaning, object, purpose, definition of the ideas and expectations of another moment, temporal region, or person. Now such an assertion, in any one case, may be regarded with scepticism, since it is, for human beings, unverifiable. Nobody experiences, in his own person or at any one time, the identity of the ideas, meanings, expectations, of yesterday and today, of himself and of another person, of Newton and of the later students of Nature who have tested what they believe to be Newton's ideas. One may, in each special case, doubt, therefore, whether the idea formed yesterday is the same in meaning as the idea tested today, whether two men mean the same by the hypotheses which they are trying to verify together, and so on. But this much seems clear: however doubtful, in the single case, any such proposition may appear, unless some such propositions are true, there is no such process as the repeated testing of the same ideas through successive processes of experience, occurring at separate moments of time, or in the experiences of various human observers. But in that case it is not true that the proposition, 'Such a testing of ideas by the course of experience as Pragmatism presupposes actually takes place,' expresses the facts. If, however, this proposition is not true, the whole pragmatist account of truth becomes simply meaningless. On the other hand, if the proposition is true, then there is a kind of truth whose nature is inexpressible in terms of the pragmatist's definition of truth. For there are propositions which no human being at any moment of his own experience can ever test, and which are nevertheless true.

Much the same may be said of the pragmatist's assertion regarding the 'workings' that an idea is said to 'possess' or to which it is said to 'lead.' These 'workings' by hypothesis, may extend over long periods of time, may find a place in diverse minds, and may involve extremely complex reasoning processes (e.g. computations, as in the case of the Newtonian theory of gravitation) which are very hard to follow, and which no human mind can survey, in their wholeness, at any moment, or submit to the test of any direct synthetic observation. The proposition, however, 'These are the actual, and, for the purposes of a given test, the logically relevant workings of the idea that is to be tested,' must itself be true, if the empirical comparison of any one of these workings with the facts of experience is to be of any worth as a test. The truth of the proposition just put in quotation marks is a truth of a type that no one man, at any instant, ever personally and empirically tests. In every special case it may be, and in general must be, regarded as doubtful.

Yet, unless some such propositions are true, Pragmatism becomes a meaningless doctrine; while, if any such propositions are true, there is a sort of truth of which Pragmatism gives no account.

What holds of truth holds here, in general, of the conditions which make falsity possible. And the whole theory of true and false opinion, and consequently the definition of error, must be modified accordingly. In brief, Pragmatism presupposes a certain unity in the meaning and coherence of experience taken as a whole --a unity which can never at any one moment be tested by any human being. Unless the propositions which assert the existence and describe the nature of this presupposed unity are themselves true, Pragmatism has no meaning. But, if they are true, Pragmatism presupposes a sort of truth whereof it gives no adequate account. To say this is not to say that Pragmatism gives a wholly false view of the nature of truth, but is only to insist upon its inadequacy. It needs to be supplemented.

(iii.) Over against the theory of truth as the correspondence between a wholly external object and an ideal portrayal, and also in contrast with Pragmatism, there exists a theory of truth which defines that concept wholly in terms of a harmony between the partial expression of a meaning which a proposition signifies and the whole of life, of experience, or of meaning, which, according to this theory, ideas and propositions intend to embody so far as they can. A proposition is true in so far as it conforms to the meaning of the whole of experience. Such conformity can never be attained through the mere correspondence of a portrayal with an external object. It can exist only in the form of the harmonious adaptation of part to whole-- an adaptation that can best be figured in the form of the adaptation of an organ to the whole of an organism.

If one reverts to the comparatively trivial instance of the sheep and the counting, the present view would insist, as Pragmatism does, upon the fact that, in counting sheep, one is attempting to adjust present ideas to the unity of an extended realm of experience, in which the observed sheep appear, now as grazing in the field, now as having their place in the herdsman's enterprises, now as passing from one ownership to another, and so on. The one who counts wants to get such a present idea of the sheep as will stand in harmonious unity with all else that can be or that is known with regard to them. The truth involved in the process of counting is itself of a relatively abstract and lower sort; and hence is ill adapted to show what truth really is. For, in fact, to treat sheep merely as numerable objects is to treat them as what, on the whole, they are not; hence to say, 'They are so many,' is to utter what is in some respects false. For they are sheep, and to say this is say that each is a living organism, a unique individual, a product of ages of evolution, and a being possessing values beyond those which commerce recognizes. Hence a numerical account of them has only 'partial truth,' and therefore is false as well as true. The only wholly true account of the sheep would express (not merely portray) their character as facts in the universe of experience and of reality. One can say, at best, of the proposition about their number that it is true in so far as it expresses a view about them, which harmonizes, to the greatest extent possible for a numerical statement, with what experience, viewed as a whole, determines the place and the meaning of one's present experience of the flock of sheep to be.

Truth from this point of view is an attribute which belongs to propositions in a greater or less degree. For single propositions, taken by themselves, give us abstract accounts of facts, or rather of the whole in which every fact has its place, and from which it derives its characters. A proposition is an interpretation of the whole universe, in terms of such a partial experience of the nature of the whole as a limited group of ideas can suggest. This interpretation is always one-sided, precisely in so far as the group of ideas in question is limited. In so far as the partial view harmonizes with the whole, the proposition is true. Since the partial view, being one-sided, can never wholly harmonize with the whole, each separate proposition, if taken in its abstraction, is partially false, and needs to be amended by adding other propositions.

This general theory of truth and falsity, while its sources run back into ancient thought, is especially characteristic of modern Idealism. That the truth of propositions about experience is a character determined by their relation to the ideal and virtual whole of experience, to the 'unity of apperception,' is a thesis which forms part of Kant's 'Deduction of the Categories.' The later developments of the 'Dialectical Method,' by Fichte and Hegel, and the analogous features of Schelling's thought, led to more explicit theories of the relations between truth and falsity, and to the doctrine that every proposition, considered in its abstraction, is partially false, and needs amendment. Hegel, in the preface to his Phänomenologie, asserted that 'Das Wahre ist das Ganze,' and interpreted this as meaning that only what a survey of the total process of experience signifies enables us to know truth, while 'partial views,' such as we get on the way towards absolutes Wissen, are at once true and false--true as necessary stages on the way to insight, and therefore as in harmony with the purposes of the whole; false, as needing supplement, and as showing this need through the contradictions which give rise to the dialectical process. In Hegel's Logic this view of truth is technically developed. With a different course of argument, with many original features, and with a more empirical method of investigation, a view of truth and error which belongs to the same general type has in recent times been developed by Bradley.

If one accepts such a theory of the 'degrees of truth and falsity,' and of truth as the harmony or organic unity between a partial view and the ideal whole of experience or of reality, the essence of error--that is, of false opinion must receive a new interpretation. In the history of the development of Absolute Idealism, the theory of error has taken, on the whole, two distinct forms.

(1) According to the first of these forms, usually emphasized by Hegel, error exists merely because it is of the essence of partial view to regard itself as the total and final view, precisely in so far as the partial view inevitably passes through the stage of 'abstraction,' in which it defines itself to the exclusion of all other points of view than its own. Did it not pass through this stage, it would not be a live or concrete view of things at all. It simply would not exist. But (according to Hegel) the whole, in order to be an organism at all, requires the parts to exist. And, if the parts are--as in the case of opinions--partial views of the whole, and if the whole requires them to exist, each in its place in the system of spiritual life, it is the whole itself, it is the Absolute, which requires the partial view to make, as it were, the experiment of regarding itself as true--that is, as an absolutely whole view. If a man is merely counting, he takes his objects simply as numerical; and then real things seem to him, as to the Pythagoreans, to be merely 'numbers.' Such a view, as an abstraction, is false; but as a stage on the way to insight it is inevitable; and as a concrete phase of opinion it is an error, that is, a positive belief in a falsity, or, again, a taking of a partial view for the whole. To be sure, this 'dwelling on the abstraction,' this beharren or verweilen in the midst of falsity, is a phase; and since, for Hegel (just as much as for the Pragmatists), the apprehension of truth is a living process, not a static contemplation, this phase must pass. An experience of the 'contradictions of finitude' must in its due time arise, and must lead to the recognition that the partial phase is false. This is what happens in the course of the history of thought, when the successive systems of philosophy--each a partial truth, required by the necessity of the thought-process and by the life of the

Weltgeist to regard itself as absolutely true--succeed one another with a dialectical necessity that tends to larger and truer insight. The same sequence of necessary errors, which are all of them partial truths taking themselves to be whole and final, appears in the history of religion.

(2) To Bradley, and to others among the more recent representatives of Idealism, to whom the dialectical method of Hegel appears in various ways unsatisfactory, this account of the way in which error arises, and, as a phase of experience and of life, is necessary, does not appeal. For such thinkers, error is, indeed defined as a partial and (in so far) false view, which is not merely partially false and partially true, but takes itself to be wholly true. The existence of such a disharmony between part and whole, in a realm of experience where the metaphysical presuppositions which these writers accept seem to require organic wholeness and harmony to prevail, and to be of the essence of reality, is an inexplicable event, which must be viewed as in some unknown way a necessary 'appearance,' not a reality.

As a statement of the ideal of truth which is alone consistent with rational demands, the Idealism thus summarized seems to be, in great measure, successful. But its success is greatest with respect to the conception of truth as the teleological harmony or adjustment of a partial to a total view of experience and of its meaning. Precisely with regard to the problem of the possibility of error, that is, of disharmony between the demands of any partial interpretation of experience and that which is revealed and fulfilled by the whole of experience, the idealistic theory of truth and of error has proved to be, thus far, most incomplete.

4. Conditions of a solution of the problem of error. The foregoing survey shows that a satisfactory theory of error must meet the following requirements:

(1) It must be just to whatever interest in a decisive and unquestionably 'absolute' distinction between true

propositions and their contradictory false propositions is justly urged by formal logic. That is, no account of truth and error in terms of 'partial views' and 'the total view' of experience must be used to render the contrast of true and false anything but a decisive contrast, as sharp as that between any proposition and its contradictory.

(2) The theory of error must take account of the actual unity of the cognitive and volitional processes. It has been the office of recent Pragmatism to insist, in its own way, upon this unity. But Hegel, in his Phänomenologie, also insisted, although in another fashion, upon the fact that every insight or opinion is both theoretical and practical, is an effort at adjustment to the purposes of life, an effort to be tested by its genuine rational success or failure.

(3) The theory of error must recognize that truth is a character which belongs to propositions so far as they express the meanings which our ideas get in their relations to experience, and not in their relations to wholly external objects.

(4) That the rational test or the success of ideas, hypotheses, and opinions lies in their relations not to momentary experiences, but to the whole of life, so far as that whole is accessible, must also be maintained.

(5) The existence of error, as disharmony between the partial view which actively and, so to speak, wilfully asserts itself as the expression of the whole, must be explained as due to the same conditions as those which make possible finite life, evil, individuality, and conflict in general.

(6) Theoretical error cannot be separated from practical error.

(7) A revision of Hegel's dialectical method, a synthesis of this method with the empirical tendencies of recent Pragmatism, a combination of both with the methods of modern Logic seem, in their combination,

to be required for a complete treatment of the problem of error. An error is the expression, through voluntary action, of a belief. In case of an error, a being, whose ideas have a limited scope, so interprets those ideas as to bring himself into conflict with a larger life to which he himself belongs. This life is one of experience and of action. Its whole nature determines what the erring subject, at his stage of experience, and with his ideas, ought to think and to do. He errs when he so feels, believes, acts, interprets, as to be in positive and decisive conflict with this ought. The conflict is at once theoretical and practical.

Editor's Note: It is the editor's opinion that no honest and disinterested thinker can deny the validity of Royce's criticism of the pragmatist and the correspondence theory of truth as set forth in this and in the preceding chapter. See R. B. Perry: *The Thought and Character of William James*, Little Brown & Co., 1935, Vol. I, pp. 778-819, and Vol. II, pp. 735, for an informative statement of the controversy between James and Royce over this and other issues, together with the text of letters they exchanged, and other written comments. See also the editor's *The Principles of Reasoning*, 3rd edition, Appleton, Century, Crofts, Inc., New York, 1947, Ch. XXVI.

Chapter V

AXIOM

Editor's Note: This article and those entitled *Error and Truth*, *Mind*, *Negation*, and *Order* were contributed by Professor Royce to the *Encyclopedia of Religion and Ethics*, edited by James Hastings.

I. Meanings of the term.--The various senses in which the term 'axiom' is used are easily confused, and require to be carefully distinguished. We may mention five senses of the term, all of which are historically important. (1) Axiom in a predominantly epistemological sense: a proposition whose truth is self-evident; an immediately certain, objective truth. (2) Axiom in a predominantly psychological sense: a proposition of whose truth the man who calls it an axiom feels a fixed persuasion, while he regards the proposition as indemonstrable, and his faith as something fundamental and, for him, necessary; a proposition held to be true with an unwavering faith. (3) Axiom in a predominantly logical sense: a first principle, which, itself not demonstrated, can be used as a basis for demonstrations. (4) Axiom in a predominantly social sense: an opinion which is, as a fact, accepted by all who are competent to understand its import. (5) Axiom in a predominantly psycho-genetic sense: an opinion which the innate constitution and the original instinctive tendencies of the mind lead us to accept, and which we therefore do not derive merely from our experience.

1. From the point of view of sense (1) all our knowledge is supposed to be either 'mediate' or 'immediate.' An axiom is a proposition known to be true, not 'mediately,' but 'immediately.' For this view, 'intuitive knowledge,' 'immediate insight,' 'direct assurance,' or 'evidence' is presupposed, as a possible form of

knowledge and of consciousness. The criterion of an axiom is said to be that, when we consider the import of a given axiomatic proposition, this state of consciousness, this direct assurance, arises, and makes wholly unquestionable the truth of the particular axiom which comes under our observation. Here the stress is laid, therefore, first upon the immediacy of the insight in question. To think the axiom, and to know it to be true, are supposed to be simply inseparable acts. The assurance or 'intuitive knowledge' in question is further regarded according to sense (1) as objective. One does not mean by the term 'axiom,' when thus used, merely to point out the fact that a given person feels sure that this axiom is true. Sense (1) implies that whoever accepts the truth of the axiom 'intuitively knows,' that is, directly observes, the perfectly objective fact that the axiom is true.

2. Sense (2) on the contrary, lays stress upon what may turn out to be the subjective necessity with which some one feels convinced of the truth of the proposition. When such a feeling of necessity attends a conviction, and when no demonstration of the truth of the conviction can be given beyond the mere observation that, so long as one conceives the meaning of the proposition, one feels thus convinced, sense (2) requires one to call the proposition an axiom. Sense (2) therefore makes the criterion of an axiom relative to the subject who feels the necessity, and who is unable to give other reason for his conviction.

Sense (1) is present in the mind of Descartes when he speaks of propositions which we 'clearly and distinctly perceive to be true.' Sense (2) is emphasized if one lays stress upon some sort of 'unswerving' and, as one conceives, necessary 'faith' or 'assurance.' Aristotle maintains that the 'principle of contradiction' is immediately evident in sense (1). But in sense (2) various subjects, appealing each to his own subjective necessity, may regard as axioms propositions which other thinkers are known to regard as false. Thus the proposition that 'water cannot turn solid' might be regarded as an axiom in sense (2) by a dweller in the tropics, who, hearing for the first time a story of frosty weather in high latitudes, rejected it as essentially incredible, and found his unbelief wholly insurmountable.

Senses (1) and (2) are often confused. The question as to the relation between objective 'evidence' and subjective 'certainty' is central in the theory of knowledge, and only a thoroughgoing sceptic will deny that there is indeed a close connexion between at least some of our 'assurances' and the objective truth. But the danger of confounding mere 'conviction' with objective 'evidence' is manifest throughout the history both of science and of religion.

3. Sense (3) makes the use of the term 'axiom' relative to a given or proposed theory or system, consisting of propositions and of reasonings. In this third sense an axiom is a proposition which is not demonstrated in the course of the development of the system in question, but is assumed or accepted at the outset, and used as a basis for demonstrations that form parts of that system. If the system in question constitutes, or is regarded as constituting, the whole of the possible system of knowledge, then the axioms in sense (3) appear as 'absolutely first principles,' since, by hypothesis, they are essential to the rational demonstration of the truths of this system, and are nowhere to be proved in the course of any investigation that we can make. But if one is explicitly confining one's attention to some more or less limited province of knowledge, or to some special system of propositions, axioms in sense (3) may be entirely relative to that special system, and are then merely the principles presupposed, used, but not demonstrated, by the system in question.

Axioms in sense (3) might therefore be neither self-evident truths nor yet necessary convictions of anybody, but merely 'assumptions' or 'postulates.' On the other hand, sense (3), in so far as it requires an axiom to be a 'first principle,' emphasizes a character which we are all especially accustomed to connect with the term, namely, that character of logical universality which a majority of axiomatic propositions are very commonly regarded as possessing. Senses (1) and (2) could be satisfied by particular, or even by individual, propositions. Thus the proposition 'I suffer,' uttered by one who has toothache, may be viewed by the sufferer either as a necessary persuasion of his own or as a 'self-evident' objective truth. Various theories of knowledge have used such 'intuitive evidence' of present experience as the very type of axiomatic knowledge. But particular propositions and reports of experience can be used as the principles of a set of demonstrations only when they are asserted along with universal propositions. And therefore at least some axioms, in sense (3) of the term, must be universal assertions. It especially belongs to sense (3) to emphasize this universal character of at least part of the axioms of any theories.

Sense (3), in contrast with, and sometimes to the exclusion of, senses (1) and (2), has been made prominent in various modern logical discussions of the principles of theoretical science. Thus, by the 'axioms' of a given mathematical theory, recent writers mean, in many cases, propositions which one uses simply as the 'fundamental hypotheses' of the theory in question (e.g., of the theory of some one of the 'non-Euclidean' or 'non-Archimedean' geometries, or of the Cantorean 'Theory of Assemblages'). One need not assert such hypotheses to be true, except in the sense that one treats them, at least provisionally, as self-consistent assumptions about a logically possible state of things, and uses them as 'principles' or as 'primitive propositions' in some statement of a theory. An axiom, in this sense, is often opposed to a theorem, which is a proposition that is shown to follow from the principles, and that is, in this sense, demonstrated in the course of the theory in question. In two different statements of a theory (e.g. in two different theoretical developments of geometry or of number-theory) decidedly different sets of 'hypotheses' or 'postulates' may be used as the axioms of the theory. In such cases what is an axiom in one statement of a theory may appear as a theorem in another statement, and conversely; and the concept of a 'first principle' becomes then relative, not merely to the theory in question, but to a particular way of stating that theory, and of showing that certain propositions follow from certain other propositions.

If one insists, as Aristotle did, upon sense (3) as applying to certain propositions which are said to form the indemonstrable principles of <u>all</u> science, so that, without these absolutely first principles, no system of knowledge whatever is possible, then indeed, <u>unless</u> one is a philosopical sceptic, one has to assert that the absolutely first principles are also axioms in sense (1). For if all science rests upon a determinate set of absolutely first principles, and if no science can demonstrate these principles, then either all science is uncertain or some principle is 'immediately evident.' Hence for Aristotle, and for those who follow his way of treating the theory of knowledge, there are propositions which are axioms both in sense (1) and in sense (3). In consequence of the Aristotelian traditions, senses (1) and (3) have therefore come to be viewed by many philosophers as actually inseparable; so that the 'first and fundamental truths' and the 'self-evident' or 'immediately known' propositions are, in discussions of the problems relating to axioms, not infrequently simply identified. But the logically important distinction between the relatively first principles of a given theory and the intuitively evident propositions (if such there be) has been brought afresh to light, especially by the modern logical investigations of scientific theories, and should never be forgotten in dealing with the topic. If a proposition is to be called an axiom both in sense (1) and in sense (3), special reasons (such, for instance, as those of Aristotle) should be advanced for asserting that this is the case. As a fact, it can never be 'self-evident' that a proposition is an axiom in sense (3); for one can ascertain that a principle is indeed a logical basis for certain demonstrations only by taking the trouble to go through the demonstrations themselves--a highly 'mediated' procedure.

4. Sense (4) uses as the criterion of an axiom the 'universal assent,' the 'consensus' of 'all rational beings,' or sometimes the consensus of all the 'competent,' of all the 'normal,' or of the wise,' or of some class of knowing beings whose common opinion in the matter is treated as the standard opinion. The criterion here in question has frequently been emphasized, and its history forms part of the long annals of the doctrine of Nature, or of 'the natural,' or of the 'Law of Nature' and the consensus of humanity' as the standard whereby both opinions and deeds are to be judged. Criterion (4) becomes an exact one only for those who hold that, as a fact of human nature, there are indeed propositions which <u>nobody</u> denies, or are indeed propositions which <u>all</u> who understand their import affirm. In practice, however, those who appeal to 'universal assent' as the warrant for an axiom usually render their criterion somewhat inexact, by the very fact that they employ this criterion in arguments directed against opponents, who, as appears, call in question either the truth, or the evidence, or the interpretation, of the axiom that is under consideration. If the opponent himself does not wholly assent, one can hardly appeal to 'universal assent' as an evidence against him, without modifying the sense in which one calls the assent 'universal.' Such modification occurs if one regards the consensus in question as that of the 'wise' or of the 'competent,' or if one insists, in a well-known polemic fashion, that 'nobody who is in his senses' doubts the supposed axiom. Thus, in practice, an axiom in sense (4) is usually conceived in some close connexion with senses (1) and (2)--the connexion being often much confused in controversy. Not infrequently a thinker first explicitly asserts that a proposition is, for himself personally, an axiom in sense (2); then he draws the conclusion that it therefore must be an axiom in sense (1); and thus he proves, by a more or less lengthy mediate course of reasoning, that the

proposition being 'immediately evident,' cannot be proved. Since, perhaps, some opponent still remains unconvinced, and declines to admit the 'immediate evidence,' the defender of the proposition in question hereupon makes use of sense (4), and now undertakes quite convincingly to silence the objector by assuring him that nobody objects to the proposition, since it is 'known to all.' Or, if the opponent even yet persists in calling attention to the 'immediately evident' truth that at least he himself objects, the defender of the axiom finally confuses sense (4) itself by a convenient definition of the 'assent of all,' whereby the opponent is excluded from the 'all' who are worthy of consideration; and hereupon the matter becomes, of course, quite clear, although not to the opponent.

Such processes have played a great part in the history of controversy. A famous example is furnished by the controversies which have been suggested by Locke's revival, in the First Book of the Essay on the Human Understanding, of the ancient questions as to whether all men possess in common a knowledge of logical, of mathematical, and of moral truths. Especially in the case of moral principles has the interest in making out whether there is any agreement amongst all men regarding the distinction between Right and Wrong been prominent in controversy ever since Locke. Numerous defenders of an axiomatic basis for morals have sought in Anthropology for the evidence that, regarding some moral opinions, all men agree and have conceived their principles as definable in terms of sense (4).

5. Finally, in sense (5) of our list, an axiom is defined by reference to the famous doctrine of 'innate ideas'. This doctrine is one which Locke's equally famous attack upon it, in the First Book of his Essay, long made central in controversy, and the partisans of innate ideas, in the various forms which this doctrine has since assumed, have frequently connected, in many often conflicting ways, senses (1), (2), and (4), and to a certain extent sense (3), with the use of the criterion for an axiom which sense (5) emphasizes.

From the point of view of sense (5) it is essential to an axiom that it should come to our consciousness by reason of the very 'constitution' or 'original nature' of the mind. Since the modern evolutionary view of the mind emphasizes the importance of our instinctive tendencies and inherited aptitudes as psychologically determining our whole intellectual life, evolutionists of the type of Spencer have been led to favour a theory of the innateness of those pre-dispositions which, when developed through our individual experience, lead us to regard some propositions as certainly true, and as true far beyond the range of our personal experience. For Spencer an axiom is, in general, an expression in an individual of the results of the 'experience of the race,' and is in so far, indeed, innate in the individual. Such a doctrine has established new connexions between senses (4), (5), and (2), and has to some extent connected senses (1) and (3) with (5).

Nevertheless, it is at least possible that an axiom in sense (5) might prove to be an actually false proposition, for the 'innate constitution of the mind' might involve one or another aptitude to believe error. In fact, an evolutionary view, closely resembling Spencer's might lead, in a thinker less optimistic about human nature than is Spencer, to the doctrine that certain instinctive tendencies, determined by evolution, are still such as to deceive the individual. Thus the innate hostility and resentfulness which form one aspect of human nature may be viewed, by an evolutionist, as a necessary result of the conditions of conflict under which humanity has developed. And such tendencies might easily lead, in a civilized man, to a belief regarded by the individual as axiomatic in sense (5), and probably also in sense (2). This belief might take the form of the principle that one ought to avenge all injuries, and to destroy, if possible, all enemies. As a fact, however, this belief, although dependent upon the very 'constitution' of the mind of one whose ancestors have lived by war and have enjoyed blood revenge, may be, and is, a false principle of ethics. Or again, a lover's beliefs about his beloved are deeply affected by the innate constitution of his mind, and may appear to him to be, not only in sense (5) but also in sense (2), axiomatic. Yet they may be in many respects false. A pessimist, such as Schopenhauer, is fond of emphasizing the innate 'illusions' which, according to him, characterize human nature. Buddhistic doctrine is equally emphatic in characterizing the most cherished and innate convictions of common sense as both logically false and morally destructive. Salvation for the Buddhist depends upon discovering axioms in sense (1) which are extremely hard to discover, so that only the Buddhas ever attain to them. But, when once seen, these axioms are for the enlightened indeed 'self-evident.' And the knowledge of them sets aside those axioms in sense (2) which are also axioms in sense (5), and which, according to Buddhism, are due to the innate deceitfulness of desire. So little, for some men, does either innateness or subjective necessity imply self-evidence and truth.

Axioms in sense (5), furthermore, need not always be axioms in sense (2); for, as partisans of innate ideas generally admit, any individual may remain unaware of some of his inherited aptitudes for conviction. On the other hand, there is no reason why a <u>new</u> assurance, or an axiom in sense (2), may not appear in the life of somebody whom revelation or a sudden growth or 'mutation' (such as may occur in the course of evolution) endows with a faith which, just because it is novel, does not constitute an axiom in sense (5).

As for senses (4) and (5), they very frequently coincide in their denotation, but need not do so. Although what 'the very constitution of the human mind determines us to believe' is <u>ipso facto</u>, 'believed by all' <u>in case</u> the constitution in question is precisely the constitution 'common to all human minds,' there is no reason why the innate might not also be the individual, the congenital <u>variation</u> of this or that mind. The individual may possess an aptitude for conviction which belongs to his 'constitution,' but which no other man or nobody who preceded him, possesses or has possessed. This is as possible as is a new individual revelation due to any other source than the inherited temperament of the individual. Prophets, Buddhas, poets, geniuses generally, have often been credited with such aptitudes for forming out of the depths of their own natures new convictions, which they have then taught to other men. On the other hand, as Locke and other empiricists have frequently insisted, those convictions which in sense (4) are more or less common to many or even to all men need not on that account be regarded as mainly determined by our innate constitution. They may be supposed to be due to experience, which moulds men to common results.

The foregoing survey shows us that the five senses of the term 'axiom' here in question are in a large measure independent of one another, so far as their logical intension is concerned, while by virtue of their various applications, now to the same, now to different sets of propositions, these five meanings of the term 'axiom' have become painfully confused in the history of controversy and of the theory of knowledge. The result is that the term 'axiom' is a very attractive and a very dangerous term, which should never be employed by a careful thinker without a due consideration of the sense in which he himself proposes to employ it.

II. History of the term.--As to the history of the term 'axiom' and of its uses, the ancient sources are above all: (i.) Aristotle's theory of the axioms as propositions conforming both to our sense (1) and to sense (3); (ii.) Euclid's actual use of his axioms in his geometry, especially in sense (3), and in union with certain propositions called 'postulates' (which were also theoretical principles in our sense (3). The treatment of the principles of science and of morals in sense (4) as principles 'known to all,' or as known to the 'wise' or to the 'competent,' has its beginnings in pre-Socratic philosophy, plays an important part in the Platonic Dialogues, and is in various special cases and passages carefully considered by Aristotle; but becomes especially prominent in the Stoical theory of knowledge and of ethics. While sense (2) plays a part throughout the history of ancient thought, it becomes especially important in Christianity and in modern discussions of the psychological aspects of the problem of knowledge. Sense (5) implied by the Platonic theory of reminiscence, but long put into the background by the Aristotelian theory of knowledge, has come to play a very great part in modern discussion. Its completest classic expression is probably the one to be found in Leibniz's Nouveaux Essais.

The later discussion of the nature, the existence, the various senses, and the use of axiomatic truths, has been dominated since 1781 by three great movements: (1) the critical philosophy of Kant; (2) the various forms of modern Empiricism, Positivism, 'Pragmatism'; (3) the modern logical investigations of the principles of science--investigations which were especially stimulated by the famous inquiry into the axioms of Euclid's geometry, and which have since extended to the whole range of the foundations of mathematics, and also to the principles of theoretical physics, and to still other branches of scientific theory.

III. Significance for modern philosophy.--In the attempt to deal with the extremely complex philosophical problems which are suggested by the foregoing five senses of the term 'axiom,' there are one or two guiding considerations which any student of the topic may well bear in mind.

(a) First, not every philosophy which tries to avoid scepticism is forced to admit the existence of axioms in sense (1). The necessity of such an admission as the sole alternative to scepticism exists, indeed, for one who holds the opinions ascribed in the foregoing sketch to Aristotle. If all science depends upon a determinate set of absolutely 'first principles' (in sense (3), then, unless these principles are also axioms in sense (1), our result would remain sceptical, for all scientific theory would lack basis. But the Aristotelian theory of scientific procedure is not the only possible one. That theory depends upon conceiving the structure of scientific theory as necessarily linear with chains of syllogisms leading from determinate beginnings to the conclusions that constitute the scientific theory. But for a thinker such as Hegel, the ideal form of the totality of scientific theory is cyclical rather than linear. Truth may be, as a whole, a system of mutually supporting truths, whose absoluteness does not depend upon any one set of first principles, but consists in the rational coherence and inevitableness of the totality of the system. To assert such a doctrine involves considerations which cannot be developed here. It is enough that such a thesis has been attempted. From this point of view there would indeed be axioms in sense (3), viz., in relation to certain partial systems, such as this or that mathematical or logical doctrine, whose theoretical development would indeed depend upon chains of deductive reasoning. And there would also be necessary truth, both in the parts and in the whole system. But there would be no absolutely first principles, and there would also be no immediate certainties--nothing, in fact, that is purely immediate in the whole system of truth. The whole would be mediated by the parts, and conversely.

(b) Second, the traditional alternative; either this proposition is self-evident or else it is dependent upon some other proposition from which it is deduced, or else it remains uncertain, does not exhaust the logical possibilities regarding the rational discovery of truth. Omitting here the complex problem as to the relation between our experience of particular facts and the general truths which our scientific theories aim at establishing, we may point out that there are propositions such that to deny them implies that they are true. As Aristotle already observed, the principle of contradiction is itself a proposition of this type. Euclid's geometry contains more than one instance where a proposition is demonstrated by showing that the contradictory of the probandum implies the truth of this probandum. The proof that this is, in fact, the case may be no easy one, and may involve elaborate meditations. But any proposition A, such that the contradictory of A implies A is, ipso facto, a true proposition, although nobody may yet have come to feel its necessity.

When we prove a proposition, however, by showing that its contradictory implies it, we do not make this proposition 'self-evident.' Nor yet do we demonstrate the proposition merely by reference to other propositions which we have to assume as prior certainties. What we find, in such cases, is not so much 'self-evidence,' as 'self-mediation' --an essentially cyclical process of developing the inter-relations which constitute the system of truth. In case, then, there are no axioms in sense (1), we need not abandon either the ideal or the hope of the attainment of rational truth.

(c) Third, axioms in sense (2) we need and use wherever and whenever we are engaged in practical activities, or are absorbed in contemplations, such as require a laying aside of the critical sense and a limitation of the business of reflexion. But the assertion 'I am sure of this' is never logically equivalent to the assertion 'This is true.' And it is no part of the business of science or of philosophy to seek, or to remain content with, merely private 'convictions' or 'persuasions,' however 'necessary' the subject feels them to be.

(d) Fourth, axioms in senses (4) and (5) interest the anthropologist, and the student of society, of history, of religion, of psychology; they can never satisfy the student of philosophy, or in particular, of logic, and of truth for its own sake.

(e) Finally, sense (3), interpreted not absolutely but relatively, so that an axiom is a principle which lies at the basis of a certain selected system of propositions, and which is not demonstrated in the course of that system, remains the sense in which the term 'axiom' is still most serviceably employed in modern theory.

Philosophy seeks not absolute first principles, nor yet purely immediate insights, but the self-mediation of the system of truth, and an insight into this self-mediation. Axioms, in the language of modern theory, are best defined, neither as certainties nor as absolutely first principles, but as those principles which are used as the first in a special theory.

Chapter VI

INDIVIDUAL

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I. Definitions.

There are four philosophical meanings of the word 'individual'. (1) A single being, as distinct either from a collection of beings or from the logical object of the general concept; a unique being; a being at least numerically distinct from all other beings. (2) A being that cannot be divided into parts to which the name of this being will apply. Thus a general name applies to a class of objects which can be divided into classes, to any one of which the class-name can still be applied, as Frenchmen, Germans, or Russians are all equally Europeans. But the name of an individual being, as for instance Socrates, cannot be applied to any of the parts, such as the hand or foot, into which Socrates can be divided. (3) An independent, separable being, capable of existing alone. (4) In ethics: A person, an individual as opposed to a corporation or a collection of men, or to a social group or organization of any kind.

The concept of the individual is at once one of the most familiar and the most difficult both in the world of common sense and in the world of philosophy. That the beings which are to be found in the world, whether inanimate objects or living beings, whether material or mental, are individuals, i. e. are distinct, singular, and unique, is a matter of common belief and report. But what constitutes individuality, or what is the principle of individuation, has been a matter of controversy both within the realm of special science and from the point of view of logical and metaphysical definition.

In logic the individual is opposed to the various kinds of universal concepts and classes, as, for example, to genus or species, or property, or accident. In psychology it has been frequently asserted that individual objects are the immediate objects of our perceptive experience; so that it has been frequently maintained, as sufficiently defining the nature of an individual being, that an individual is any object perceived by the mind as an external or as a real object. From this point of view the idea of individuality as a character of beings is derived directly from experience, and is irreducible to simpler ideas. As against this view, it has been maintained that the direct object of our perceptual experience is always a series or complex of definable or indefinable qualities, characters, relations, and behaviours of objects. Thus we observe than an orange is coloured, is round, is of a given weight, is in given relations to other observed objects, as for instance lying on a plate. But to observe these characters, however great their number and complexity, is still to observe characters that might be shared by other oranges, and frequently by other objects, so that it seems as if the individuality of the object were precisely what we least directly perceive. In case of individual persons, it is plain that we mean by personal identity a character which could never be made the object of any simple perceptive act; and in general the concept of individual being, in the various special sciences, as well as in metaphysical inquiry, shows many characters that cannot be derived from mere perception in any case. It is somewhat easier to regard the idea of individuality as an ultimate and indefinable idea, without giving any ground whatever, in experience or in articulate thought, for our overwhelming assurance that the objects with which we deal are individuals. Nevertheless, this way of treating the problem would be merely an abandonment of the question as insoluble; and accordingly we find in the history of thought a considerable number of efforts to define what has been called the principle of individuation. Such efforts have also played a part in the definitions of the individual that have been framed to meet the exigencies at various special times.

II. History of the Principle of Individuation.

The problem of the nature of individuality is first reached in European philosophy as the result of those efforts at the definition of the logical method which assume definite form in the Socratic, Platonic, and Aristotelian doctrines. The Platonic ideas were beings supposed expressly to correspond to our general conceptions. A discussion of the nature of these ideas, and of their relations to one another and to the world of facts, of sense, and of ordinary experience, made prominent the question: What is meant by an individual? The problem in question is very explicit in the mind of Aristotle, though it cannot be said that he gave any very explicit solution of the questions that he himself raised. According to Aristotle, all beings in the universe are individuals, and universals have being only as realized in individual entities. On the other hand, science, according to Aristotle, is necessarily concerned with the universal, that is, with the laws and the ideal general characters possessed by facts. And yet, according to Aristotle, science undertakes to know being. Thus arises the familiar paradox of the Aristotelian system, that just that character of facts which science best knows is not that character which constitutes the true being of anything, since this true being is individual, and what science knows is universal. As to the principle of individuation, Aristotle is not explicit, except in the case where he is speaking of obviously corporeal entities, when he upon occasion says that their material aspect is that which constitutes the individuality of any one being; while what he calls the form or general nature is common to many individuals.

The problem of definition of the individual became prominent in the philosophy of the Christian Church in the Middle Ages, especially owing to the importance which ethical individuality had acquired in Christian doctrine, and in consequence of the relation of individuality to the doctrine of the Trinity. In the early ages of scholastic philosophy, the discussion regarding the nature of universals was especially prominent, while from the age of Thomas Aquinas onwards, scholastic philosophy made especially important the principle of individuation and the nature of individuality. Here it was a classic Thomistic doctrine that, in the realm of nature, matter is the individuating principle, so that the purely incorporeal entities, such as the angels, could be individuated only by their form, and so that consequently, as the well-known Thomist theory maintained, two angels of the same species do not exist. The ethical individuals in the human world, according to Thomas, are genuine individuals, but they are individuated primarily by the different bodies to which the souls belong, so that when absent from the body, the soul, between death and the judgment, would be individuated by reason of what Thomas calls its _inclinatio_ to a particular corporeal embodiment, an _inclinatio_ which at the resurrection would be met by the presence of the glorified body of the saved or the equally permanent material organization of the lost. To the Thomistic Theory, Duns Scotus opposed the doctrine that a special form, called by him the _haecceitas_, is responsible for the individuation of every being, corporeal or incorporeal. This _haecceitas_, different in each individual, is something that is said to be 'fused with the common nature,' in such wise that the doctrine of Duns Scotus especially lays stress upon the fact that the difference between individual beings is a real and ultimate, non-corporeal, but not necessarily indescribable, character of each being--a character which a higher type of intelligence than our own may be able to appreciate, although it is admitted by Duns Scotus that this character is indefinable for our own human intellect.

In the philosophy of Leibnitz an effort is made to solve the problem of the individual by the famous principle of the identity of indescernibles. According to this principle it is impossible that two individuals in the universe could be precisely alike. And the unlikeness between individuals is always of an essentially describable or intelligible character. Were two individuals in all respects precisely alike, so that whatever was predicated of one was true of the other, these two individuals would be ipso facto identical, and difference of individuality means difference of quality or character.

Post-Kantian thought, during the idealistic period, laid most stress upon the definition of the ego, or of the ethical and epistemological person, and dealt with the problem of individuality mainly in this case. New in this period of thought, therefore, especially after Kant's theory of the unity of self-consciousness, are the Fichtean and Schellingean attempts to define the relations between a finite individual and the absolute ego. New is also the Hegelian definition of the individual as 'the unity of the universal and the particular.' In other words, Hegel held that when a universal law or principle, of the type that he defined as the Begriff, gets a complete development and expression, in respect of all the particular or specific aspects which the nature of this Begriff involves, such total expression of a universal law is as such an individual being, so that the problem with regard to the individual becomes identical with the problem as to the way in which universal principles can find complete or wholly satisfying expression in nature or in mind, or in general in experience.

Since the close of the idealistic period the problem of the individual has received discussion especially from three points of view. In the first place, in modern psychology and epistemology, efforts have been made to deal afresh with the problems of the nature of individuality in its most general form. The most familiar of these efforts is one not wholly unknown to earlier thought, very plainly stated by Schopenhauer, and frequently developed in recent works--an effort to make time and space the essential principles of individuation, to define the primary individual object as that which is in a given place, and at a given time, and to make the other forms of individuality derivable from this primary form of the idea by means of various associations between time and place, localization, and the various other characters possessed by moral or other types of individuals. The second point of view from which the problem has been discussed has been that of the biological sciences, and of the related branches of inquiry. Into these discussions this is not the place to go. The third point of view with regard to the individual has been that suggested by the ethical problem of the rights of the individual as a person, and of his relations to the social order. The revolt against certain eighteenth-century doctrines concerning the rights and position of the individual man has led to a number of forms of socialism and of ethical universalism, which have made light of the historical importance and of the moral value of the individual man. A frequent reaction against these very tendencies has led to a reassertion of individualism, which has been associated with various more or less novel efforts to restate the definition of the ethical individual.

III. Meaning of Individual in the Sciences.

If we survey the problem of the individual apart from its history, it is easy to see that the question has several distinct forms, which may indeed be ultimately connected, but which are usually presented to our attention in quite different regions of inquiry. Amongst the sciences mathematics is prominent in dealing with individual objects, and systems of individual objects, which as artificial creatures of definition, or as very simple abstractions from our experience of space, ought apparently to be topics of easy and final agreement. Yet regarding the nature of all these forms of mathematical individuality considerable differences of opinion has existed amongst mathematical experts. Examples of mathematical entities that appear more or less obviously as individuals are the unit in arithmetic, the point in geometry, the line, when regarded as an elementary concept in some forms of geometrical investigation, and several other cases of objects which are regarded as the elements of given mathematical systems. The question of the individuality, or at any rate of the singularity of such objects as space and time, viewed in their wholeness, has interested the mathematicians as well as the philosophers. In theoretical natural science, such concepts as the modern concept of energy, when compared with the usual concepts of matter, may well introduce the question whether recent theory is not really as much concerned with the problem whether universals exist, or whether individuals alone are real, as was ever scholastic doctrine. For energy as usually described appears to be an entity whose individuation is altogether problematic, and whose known character seems to be entirely of universal type. In the biological sciences the problem as to the living individual introduces entirely different questions and interests; and the problems of ethical individuality belong to still another realm of decidedly special character. Finally, the problem of the ultimate place of the category of individuality in the world at large remains as an issue for general metaphysics. It is, nevertheless, a fair question for philosophical inquiry whether all these so various problems are not really much more closely connected than they seem, and whether a final definition which will hold for all forms of individuality may not yet be discovered.

Chapter VII

MIND

Editor's Note: See the editor's note above under *Axiom*, p. 125.

The present article must be limited to a discussion of the metaphysical theories of mind. Owing to the peculiar position which these problems occupy in philosophy, as well as in the study of ethical and religious problems, it is advisable, first of all, to make explicit some of the epistemological problems which especially confront the student of the nature of mind; and in order to do this, we must, in view of numerous traditional complications which beset the theory of the knowledge of mind, open our discussion with some general statements concerning the nature of problems of knowledge.

The history of epistemology has been dominated by a well-known contrast between two kinds of knowledge, namely, perceptual knowledge and conceptual knowledge. This dual contrast seems insufficient to supply us with a basis for a really adequate classification of the fundamental types of knowledge. It is proposed in the present article to base the whole discussion upon a threefold classification of knowledge. Having begun with this threefold classification and briefly illustrated it, we shall go on to apply it to the special problems which we have to face in dealing with mind. We shall then consider in some detail what kinds of mental facts correspond to the three different kinds of knowledge thus defined. In conclusion, we shall deal with some problems of the philosophy of mind in the light of the previous discussion.

I. Perception and conception as fundamental cognitive processes.—A careful study of the processes of knowledge, whether these occur in the work of science or in the efforts of common sense to obtain knowledge, shows us three, and only three, fundamental processes which are present in every developed cognitive activity and interwoven in more or less complicated fashion. Of these two have been recognized throughout the history of science and philosophy, and their familiar contrast has dominated epistemology. The third, although familiar and often more or less explicitly mentioned, was first distinguished with sharpness, for epistemological purposes, by the American logician, Charles Peirce. We shall speak first of the two well-known types of cognitive process, perception and conception.

The name 'perception' is used in psychology with special reference to the perceptions of the various senses. We are here interested only in the most general characteristics of perception. William James has used, for what is here called perception, the term 'knowledge of acquaintance.' He distinguishes 'knowledge of acquaintance' from 'knowledge about.' In the simplest possible case one who listens to music has 'knowledge of acquaintance' with the music; the musician who listens in the light of his professional knowledge has not only 'knowledge of acquaintance,' but also 'knowledge about'; he recognizes what changes of key take place and what rules of harmony are illustrated. A deaf man who has learned about the nature of music through other people, in so far as they can tell him about it, but who has never heard music, has no 'knowledge of acquaintance,' but is limited to 'knowledge about.' 'Knowledge of acquaintance' is also sometimes called 'immediate knowledge.' In the actual cognitive process of the individual human being it never occurs quite alone, since, when we know something perceptually or by acquaintance, we also always have more or less 'mediate' knowledge, i.e., one who listens to music, but who also considers the person of the artist, the relation of the music to the programme, the name of the composer, or the place of this experience in his own life, has in his knowledge that which is more than the immediate hearing of the music.

'Knowledge about' includes, on occasion, mental processes which may vary very widely and which may be mingled with 'knowledge of acquaintance' in ways which are far too complex to analyze here. But 'knowledge about' is especially opposed to 'knowledge of acquaintance' in one class of cases which need to be emphasized through the use of a special name. We may name that class by calling the kind of knowledge involved in it by the name already used, 'conceptual knowledge.' Conceptual knowledge is knowledge of universals, of relations, or of other such 'abstract' objects. The Socratic-Platonic theory of knowledge called attention from its very beginning to universals and relations, and consequently made this type of knowledge specially prominent.

No doubt, even if one is disposed to cling to this merely dual classification of knowledge, one may well question whether all knowledge which is not merely 'knowledge of acquaintance' is of the grade of conceptual knowledge. For there is much 'knowledge about' concerning which we should all hesitate to say that it is knowledge of universals. Socrates himself, in his effort to define the knowledge of universals, met at the start with the fact that much of our knowledge of universals is confused and inarticulate. But if, for the moment, we neglect the intermediate cognitive states in which we more or less mingle 'knowledge of acquaintance' and conceptual knowledge, or possess conceptual knowledge in imperfect degrees of development, we may readily admit that this traditional dual classification of cognitive states is sufficient to call attention to a distinction which is of the utmost importance, both for empirical science and for metaphysics.

While the distinction between perceptual and conceptual knowledge is of great importance in determining the distinction between the deductive and the inductive methods in the sciences, the classification of these two modes of cognition does not of itself suffice to determine what constitutes the difference between inductive and deductive science. When we have clear and accurate conceptual knowledge, we are in general prepared to undertake scientific processes that in the case of further development will involve deductive methods. Thus, in particular, a conceptual knowledge of universals leads, in the mathematical sciences, to the assertion of propositions. Some of these propositions may appear at the outset of a science as axioms (see the chapter on AXIOMS above.) Whether accepted as necessarily true or used merely as hypotheses, these propositions, either alone or in combination, may, and in the mathematical sciences do, form the starting-point for a system of rational deductions. The type of knowledge involved in this deductive process will be, in the main, the conceptual type. In what sense and to what degree a 'knowledge of acquaintance' enters into a process of methematical reasoning we have not here to consider. All will admit that the sort of knowledge which dominates such a deductive process is 'abstract,' is concerned in reaching results which are true about the propositions that themselves form the premisses of the deduction. And so our knowledge concerning numbers, the operations of a mathematical science, and similar cases form exceptionally good instances of what characterizes conceptual knowledge in its exact and developed form.

In the inductive use of scientific methods we find a more complicated union of the perceptual and the conceptual types of knowledge. When a hypothesis, such as Newton's formula for gravitation, or Galileo's hypothesis concerning the laws of falling bodies, is stated, the type of knowledge involved in formulating and in understanding the hypothesis is prevailingly conceptual. When the hypothesis is tested by comparing the predictions based upon it with experience, the test involves appealing at some point to perceptual knowledge, or 'knowledge of acquaintance.' The processes of experiment used in an inductive science might seem to be typical cases of processes involving perceptual knowledge. And experiments unquestionably do involve such knowledge. But an experiment reveals a truth, because it brings concepts and percepts into some sort of active synthesis. Upon such active synthesis depends the process of validation which is used as the basis for the definition of truth used by recent pragmatists (see the chapter on ERROR AND TRUTH above).

In so far as we insist upon this dual classification of fundamental processes of cognition, the questions which most come to our notice, regarding both knowledge and its objects, concern (1) the relative value of these two cognitive processes, and (2) the degree to which, in our actual cognitive processes, or in ideal cognitive processes (such as we may ascribe to beings of some higher order than ours), the two can ever be separated. These two questions have proved especially momentous for the theory both of knowledge and of reality.

(1) Regarding the relative value of the two fundamental types of cognition, Plato, as is well known, held that conceptual knowledge is the ideal type, the right result an expression of reason. Conceptual knowledge gives truth; perceptual knowledge gives illusion or appearance--such is, on the whole, the Platonic doctrine. In recent discussion the pragmatists--and still more emphatically Bergson--have insisted upon the relative superiority of the perceptual type of knowledge. The familiar expression of this view is the thesis of recent pragmatism that conceptual knowledge has only a sort of 'credit value'; perceptual knowledge furnishes the 'cash of experience'; conceptions are 'bank notes'; perceptions, and perceptions only, are 'cash'. The statement of Bergson goes further, and declares that, if we had unlimited perceptual knowledge, i.e. 'knowledge of acquaintance' whose limits and imperfections we had no occasion to feel, because it had no limits and no imperfections, then conceptions could have no possible interest for us as cognitive beings. In other words, we use concepts, i.e., we seek for a

knowledge of universals, only when our perceptions in some way fail us. Conceptual knowledge is in its very essence a substitute for failing perceptual knowledge. The opposition between Plato and Bergson regarding this estimate of the relative significance and truthfulness of the two kinds of cognitive processes is thus characteristic of the contrast which is here in question. Of course all the philosophers admit that, in practice, our knowledge makes use of, and from moment to moment consists in, a union which involves both conceptual and perceptual processes.

(2) On the question whether the two foregoing types of knowledge, however closely linked in our normal human experience, can, at least in ideal, be separated--i.e., whether a knowledge by 'pure reason' is possible on the one hand, or a knowledge of 'pure experience' is ever attainable on the other hand--the historical differences of opinion are closely related to well-known metaphysical controversies. For Plato, as (in another age, and in a largely different metaphysical context) for Spinoza, it is at least in ideal possible for philosophy, or for the individual philosopher, to attain a purely intellectual insight into the realm of 'ideas' or into the nature of the 'substance.' For various forms of mysticism, as well as for theories such as the one set forth in the Kritik der reinen Erfahrung (Leipzig, 1888-90) of R. H. L. Avenarius, a mental transformation may be brought about through a process which involves either a practical or a scientific correction and gradual suppression of erroneous intellectual illusion; and, at the limit of this process, reality becomes immediately and perceptually known, without confusion through abstractions.

The 'radical empiricism' of James's later essays makes use of a theory of knowledge which attempts, as far as possible, to report, apart from conceptual constructions, the data of pure experience.

2. Interpretation through comparison of ideas as a third fundamental cognitive process.--It is an extraordinary example of a failure to reflect in a thoroughgoing

way upon the process of knowledge that until recently the third type of cognitive process to which we must next refer has been neglected, although every one is constantly engaged in using and in exemplifying it.

When a man understands a spoken or written word or sentence, what he perceives is some sign, or expression of an idea or meaning, which in general belong to the mind of some fellow-man. When this sign or expression is understood by the one who hears or who reads, what is made present to the consciousness of the render or hearer may be any combination of perceptual or conceptual knowledge that chances to be in question. But, if any one cries 'Fire!', the sort of knowledge which takes place in my mind when I hear and understand this cry essentially depends upon this fact; I regard my fellow's cry as a sign or expression of the fact either that he himself sees a fire or that he believes that there is a fire, or that, at the very least, he intends me to understand him as asserting that there is a fire, or as taking an interest of his own in what he calls a fire. Thus, while I cannot understand my fellow's cry unless I hear it, unless I have at least some perceptual knowledge, and while I equally shall not have a 'knowledge about' the nature of fire, and so a 'knowledge about' the object to which the cry refers, unless I am possessed of something which tends to be conceptual knowledge of his object, my knowledge of my fellow's meaning, my 'grasping of his idea,' consists neither in the percept of the sign nor in a concept of its object which the sign arouses, but in my <u>interpretation</u> of the sign as an indication of an idea which is distinct from any idea of mine, and which I refer to a mind not my own, or in some wise distinct from mine.

It is to be noted that, however we reach the belief in the existence of minds distinct from our own, we do not regard these minds, at least in ordinary conditions, as objects of our own perceptual knowledge. For the very motives, whatever they are, which lead me to regard my perceptions as my own even thereby lead me to regard my fellows' perceptions as never present within my own field of awareness. My knowledge of my own physical pains, of the colours that I see, or of the sounds that I hear is knowledge that may be called, in general terms, perceptual. That is, these are objects with which I am, or upon occasion could be, acquainted. But with my fellow's pains I am not acquainted. To say this is merely to say that, whatever I mean by 'myself' and by 'the Alter,' the very distinction between the two is so bound up with the type of cognition that is in question that whatever I am acquainted with through my own perception is ipso facto my own object of acquaintance. Thus, then, in general, perceptual knowledge has not as its object what is at the same time regarded as the state of another mind than my own.

But, if the mind of my fellow, in particular his ideas, his feelings, his intentions, are never objects of perceptual knowledge for me, so that I am not directly acquainted with any of these states, must we regard our knowledge of the mind, of the ideas, of the intents, purposes, feelings, interests of our fellow-man as a conceptual knowledge? Is our fellow-man's mind the object of a concept of our own? Is the fellow-man a universal, or a relation, or a Platonic idea? Wherein does he differ from a mathematical entity or a law of nature? Unquestionably we regard him as possessing conceptual knowledge of his own, and also as engaged in processes of knowledge which may be conceptual, or which may involve any union of percept and concept. But the fact remains that neither by our own perceptions can we become acquainted with his states of mind, nor yet by our own conceptions can we become able to know the objects which constitute his mental process. In fact, we come to know that there are in the world minds not our own by interpreting the signs that these minds give us of their presence. This interpretation is a third type of knowledge with is closely interwoven with perceptual and conceptual knowledge, very much as they in turn are bound up with it, but which is not reducible to any complex or combination consisting of elements which are merely perceptual or merely conceptual.

Every case of social intercourse between man and man, or (what is still more important) every process of inner self-comprehension carried on when a man endeavours to 'make up his own mind' or 'to understand what he is about', involves this third type of cognition, which cannot be reduced to perception or to conception. It is to this third cognitive process that, following the terminology which Peirce proposed, we here apply the name 'interpretation.'

In order to distinguish more clearly the three types of cognition, we may say that the natural object of perception is some inner or outer datum of sense or feeling, such as a musical tone, a colour, an emotional state, or the continual flow of the inner life upon which Bergson so much insists. For these are typical objects of perceptual knowledge, *i.e.* of 'knowledge of acquaintance.' The typical objects of conceptual knowledge are such objects as numbers, and relations such as identity and difference, equality, and so on. But typical objects of interpretation are signs which express the meaning of some mind. These signs may be expressions of the meaning of the very mind which also interprets them. This is actually the case whenever in memory we review our own past, when we reflect upon our own meaning, when we form a plan, or when we ask ourselves what we mean or engage in any of the inner conversation which forms the commonest expression of the activity whereby an individual man attains some sort of explicit knowledge of himself.

The form of cognitive process involved in the social relations between man and man is essentially the same a as that involved in the cognitive process by which a man makes clear to himself his own intent and meaning. For, despite well-known assertions to the contrary on the part of Bergson, nobody has any adequate intuitive 'knowledge of acquaintance' with himself. If such perceptual or intuitive knowledge of the self by the self were possible, we should not be obliged to acknowledge that the world of human beings is dominated by such colossal and often disastrous ignorance of every man regarding himself, his true interests, his real happiness, his moral and personal value, his intents, and his powers, as we actually fine characterizing our human world. In brief, man's knowledge, both of himself and of his neighbour, is a knowledge which involves an interpretation of signs. This thesis, very ably maintained by Peirce in some of his early essays, involves consequences which are at once familiar and momentous for the theory of knowledge.

That the type of knowledge involved whenever signs are interpreted is a fundamental type of knowledge which cannot be represented either to perception or to conception can be exemplified in most manifold ways, and will appear somewhat more clearly through the illustrations given below. It may be useful to point out here that, while all our interpretations, like all our perceptual and conceptual knowledge, are subject to the most manifold illusions in detail, it still remains the case that, whenever one is led to attempt, propose, or believe an interpretation of a sign, he has actually become aware, at the moment of his interpretation, that there is present in his world some meaning, some significant idea, plan, purpose, undertaking, or intent, which, at the moment when he discovers its presence, is from his point of view not identical with whatever idea or meaning is then his own.

If somebody speaking to me uses words which I had not intended to use, I may misunderstand the words, or I may not understand them at all. But, in so far as I take these words to be the expression of a meaning, this meaning is one that just then I cannot find to be my own--*i.e.*, these words do not express my ideas, in so far as these ideas are by me interpreted as my own. The cognitive process here in question divides, or at least distinguishes, that part of the objects, ideas, or meanings in question into two distinct regions, provinces, or modes of mental activity. One of these regions is interpreted at the moment as 'my own present idea,' 'my own purpose,' 'my own meaning'; the other is interpreted as 'some meaning not just now my own,' or as 'some idea or meaning that was once my own'--i.e., as 'my own past idea,' or as 'my neighbour's meaning,' or perhaps as 'a meaning that belongs to my social order,' or 'to the world,' or, if I am religiously minded, 'to God.' In each case the interpretation that is asserted may prove to be a wrong one. Interpretation is fallible. So, too, is conception, when viewed as a cognitive process, and so is perception, whose character as 'acquaintance with' is no guarantee of its accuracy, whether mystical apprehension or ordinary observation is in question. The fact for our present purpose is not that our human knowledge is at any point infallible, but that there is the mode or type of cognition here defined as interpretation. Interpretation is the knowledge of the meaning of a sign. Such a knowledge is not a merely immediate apprehension, nor yet a merely conceptual process; it is the essentially social process whereby the knower at once distinguishes himself, with his own meanings, ideas, and expressions, from some other self, and at the same time knows that these selves have their contrasted meanings, while one of them at the moment is expressing its meaning to the other. Knowledge by interpretation is, therefore, in its essence neither mere 'acquaintance' nor yet 'knowledge about.'

There is another way of expressing the distinction of these three kinds of knowledge which proves useful for many purposes. Knowledge of the first kind, 'knowledge of acquaintance,' may for certain purposes be characterized as 'appreciation.' Conceptual knowledge, owing to the means often employed in making a concept explicit, may be for many purposes called 'description.' In each case, as will be noted, the main character of the type of knowledge in question can be designated by a single term, namely, appreciation or description, just as in the foregoing these two types of knowledge have been designated each by a single term, acquaintance in one case and conception in the other.

In designating the instances of interpretation it is well to note that every interpretation has three aspects.

For the one who interprets it is an expression of his own meaning. With reference to the object, i.e. to the sign, or to the mind whose sign this is, the interpretation is the reading or rendering of the meaning of this mind by another mind. In other words, every interpretation has so far a dual aspect: it at once brings two minds into quasi-social contact and distinguishes between them or contrasts them. In the light of this contrast and with reference to the direction in which it is read, the two minds are known each in the light of the other. As has already been said, the two minds in question may be related as a man's own past self is related to his present or future self. And in fact, as Peirce has pointed out, every act of interpretation has also a triadic character. For the cognitive process in question has not only a social character, but what one may call a directed 'sense.' In general, when an interpretation takes place, there is an act B wherein a mental process A is interpreted, read, or rendered to a third mind. That the whole process can take place within what, from some larger point of view, is also a single mind with a threefold process going on within it has already been pointed out. Thus, when a man reflects on his plans, purposes, intents, and meanings, his present self, using the signs which memory offers as guides, interprets his past self to his future self, the cognitive process being well exemplified when a man reminds himself of his own intents and purposes by consulting a memorandum made yesterday for the sake of guiding his acts to-day. Every explicit process involving self-consciousness, involving a definite sequence of plans of action, and dealing with long streches of time, has this threefold character. The present self interprets the past self to the future self; or some generally still more explicit social process takes place whereby one self or quasi-self has its meanings stated by an interpreter for the sake of some third self.

Thus, in brief, knowledge by interpretation is (1) an expression (by an 'interpreter') of (2) the idea or meaning whereof some other mind gives a sign, and (3) such an expression as is addressed to some third mind, to which the interpreter thus reads or construes the sign.

3. Self-interpretation, comparison of one's own ideas, and knowledge of time.--When such interpretation goes on within the mind of an individual man, it constitutes the very process whereby, as is sometimes said, he 'finds himself,' 'comes to himself,' 'directs himself,' or 'gets his bearings,' expecially with reference to time, present, past, and future. In the inner life of an individual man this third mode of cognition, therefore, appears at once in its most fundamental and simplest form as the cognitive process whose being consists in a comparison of ideas. The ideas compared here belong in one sense to the 'same self'; but they differ as the ideas of 'past self' and 'future self'; or, in various other ways, they belong to different 'quasi-minds.'

That such a process is, indeed, irreducible to pure perception, to pure conception, or to that active synthesis of the two which James has in mind when he uses the term 'idea,' readily becomes manifest if we consider what takes place when two 'ideas' are 'compared,' whether these two belong to men who are 'different individuals' or to the past, present, or future selves of one who is, from another point of view, the same man.

An 'idea,' when the term is used in the sense which recent pragmatism[1] has made familiar and prominent, is not a mere perception, nor a mere collection or synthesis of various perceptions, images, and other immediate data; nor yet is it a mere conception, whether simple or complex. It is, for James and his allies, a 'leading,' an 'active tendency,' 'a fulfilment of purpose,' or an effort towards such a fulfilment, an

[1]See W. James, Pragmatism, London, 1907

'adjustment to a situation,' a seeking for the 'cash,' in the form of sense-data, such as may, when found, meet the requirements, or 'calls,' made by the conceptual aspect of the very idea which is in question. This concept has, in Bergson's phrase, its 'credit value.' Eventual sense-data may furnish the corresponding 'cash.' The idea is the seeking for this 'cash.' When the wanderer in the woods decides to adopt the idea that 'yonder path leads me home,' he makes an active synthesis of his concept of home and of his present sense-data. This active synthesis expressed in his idea, 'I am homeward bound,' is a 'leading,' which, if he is successful, will result in furnishing to him, when his wanderings cease, the perceptions of home which constitute the goal of his quest. This, then, is what is meant by the term 'idea' in that one of its senses which pragmatism has recently most emphasized.

In this way we may also illustrate how the cognitive process possesses the two forms or aspects which have usually been regarded as the only fundamentally distinct aspects of knowledge: perception and conception. We meanwhile illustrate that active <u>union</u> of these two which constitutes the 'idea' as defined by recent pragmatism. But we do not thus illustrate an aspect of cognition which is equally pervasive and significant, and which consists in the <u>comparison of ideas</u>. It is just this aspect of cognition upon which our present theory most insists. For by what process does the wanderer, when he reaches home, recognize that this home which he finds is the very home that he had sought? Not by the mere presence of a 'home-feeling,' not by a perception which, merely at the moment of home-coming, pays the 'cash' then required by some then present conception of home, but by a process involving a comparison of his ideas about his home, at the moment when he reaches home, with his memories of what his ideas were while he was lost in the woods and while he still inquired or sought the way home.

In order to consider what such a comparison essentially involves, it is not necessary to suppose that the act of comparison must take place in a form involving any high grade of self-consciousness, or depending upon a previous formation of an elaborate system of ideas about the self, the past, and similar objects. The essentially important fact is that whoever begins, even in the most rudimentary way, to take account of what seems to him as if it were his own past, whoever is even vaguely aware that what he has been seeking is the very object which now he finds, is not merely perceiving the present, and is not conceiving the past, and is not simply becoming aware of his present successes and disappointments as present facts--he is comparing his ideas of present success and failure with his ideas of his past efforts. This comparison is essentially an interpretation of some portion of his own past life, as he remembers that life, in the light of his present successes or disappointments, as he now experiences them. A third cognitive process is then involved. This interpretation compares at least two ideas: (1) the past idea or 'leading' (e.g., the past search for home by the path through the woods); (2) the present success or failure (e.g., the reaching home itself, or getting to the close of some stage of the wandering); and, in making this comparison, this interpretation estimates the result, perhaps in the light of one's idea of one's own future ('and henceforth I need search no more'), or perhaps in the light of one's idea of one's entire self ('I have succeeded,' or 'I am a knower of the truth,' or 'So much of the world of reality is mine'). In any case two comments may be made upon every such act of comparing two ideas and interpreting one in the light of the other.

(1) Unless such processes of comparing ideas were possible, and unless, in at least some rudimentary form, it took place, we could never make even a beginning in forming a coherent view of our own past and future, of our own selves as individuals, or of selves not our own. Our ideas both of the Ego and of the Alter

depend upon an explicit process of comparing ideas. The simplest comparison of ideas--such as the case upon which recent pragmatism lays so much stress--the comparison upon which the very idea 'my success' also depends, the comparison, namely, which is expressed by saying, 'What I sought at a past moment is the very same as what, at the present moment, I now find,' is an instance of an act of interpretation, and is not reducible to the two other types of knowledge.

(2) All such processes of comparison are equally characteristic of the cognitive activity which goes on during our explicitly and literally social life and of the cognitive activity which is needed when we think about our relations to our own individual past and future. In brief, neither the individual Ego nor the Alter of the literal social life, neither past nor future time can be known to us through a cognitive process which may be defined exclusively in terms of perception, of conception, and of the ideal 'leadings' of the pragmatists. The self, the neighbour, the past, the future, and the temporal order in general become known to us through a third type of cognition which consists of a comparison of ideas--a process wherein some self, or quasi-self, or idea interprets another idea, by means of a comparison which, in general, has reference to, and is more or less explicitly addressed to, some third self or idea.

4. <u>The relation of the three cognitive processes to our knowledge that various minds exist and to our views about what sorts of beings minds are.</u>--The use of the foregoing classification of the types of cognitive processes appears of special importance as soon as we turn to a brief outline of some of the principal theories about the nature of mind which have played a part in the history of philosophy. Nowhere does the theory of knowledge show itself or more importance in preparing the way for an understanding of metaphysical problems than in the case of the metaphysics of mind. No attentive student of the problem of mind can easily fail at least to feel, even if he does not very

explicitly define his feeling, that in dealing with the philosophy of mind both common sense and the philosophers are accustomed to combine, sometimes in a very confused way, a reference to different more or less hypothetical beings, while the ideas that are proposed with regard to the nature of these beings are of profoundly different types.

Thus it may be a question for common sense or for a given metaphysical doctrine as to whether or not there exists a so-called soul. Now it makes a great difference for the theory of the soul whether the kind of soul which is in question is viewed as in its essence an object of a possible immediate acquaintance or perception, as an object of a possible adequate conception, or as an object whose being consists in the fact that it is to be interpreted thus or so. Unless the three kinds of cognition are clearly distinguished, the one who advances or tests a given theory of the soul does so without observing whether he himself is speaking of the soul as a possible perception, or is treating it as if it were, in its inmost nature, an object which can be known only through some adequate conception. If one has called to his attention the fact that he is speaking now in perceptual and now in conceptual terms of the mind or soul which his theory asserts to be real, he may then attempt to solve his difficulties in the way which recent pragmatism has emphasized, i.e., he may declare that his doctrine is of necessity a 'working hypothesis' about the nature of the soul, that it is, of course, in part stated in conceptual terms, but that the concepts are true only in so far as they prove to be somewhere directly verifiable in terms of immediate percepts.

Yet nowhere does recent pragmatism, in the form in which William James left it, more display its inadequacy as a theory of knowledge than in the case where it is applied to an effort to define the truth of hypotheses concerning mind, or to test such truth. For, as a fact, nobody who clearly distinguishes his neighbour's individual mind from his own expects, or can consistently anticipate, that his neighbour's mental states, or that anything which essentially belongs to the inner life or to the distinct mind of his neighbour, can ever become, under any circumstances, a direct perception of his own. For, if my neighbour's physical pains ever became mine, I should know them by immediate acquaintance only in so far as they were mine and not my neighbour's. And the same holds true of anything else which is supposed to be a fact essentially belonging to the individual mind of my neighbour. At best I can hope, with greater or less probability, to interpret correctly the meaning, the plan, or some other inner idea of the mind of my neighbour; but I cannot hope to go beyond such correct interpretation so far as to perceive my neighbour's mental states. For, if my neighbour's mental states became the immediate object of my own acquaintance, my neighbour and I would so far simply melt together, like drops in the ocean or small pools in a greater pool. The immediate acquaintance with my neighbour's states of mind would be a knowledge neither of himself as he is in distinction from me nor of myself as I am in distinction from him. For this general reason 'working hypotheses' about the interior reality which belongs to the mind of my neighbour can never be 'converted into the cash of experience.' My neighbour's mind is never a verifiable object of immediate acquaintance, precisely as it is never an abstract and universal idea. The one sort of knowledge for which recent pragmatism has no kind of place whatever is a knowledge, statable in pragmatistic terms, concerning my neighbour's mind.

James himself follows a well-known and ancient philosophical tradition by declaring that our assertion of the existence of our neighbour's mind depends upon the argument from analogy. Because of similar behaviours of our organism we regard it as by analogy probable that both our neighbour's organism and our own are vivified by more or less similar mental lives, so that we have similar experiences. But to regard or to believe in the mind of our neighbour as an object whose existence is to be proved through an argument from analogy raises a question whose answer is simply fatal to the whole pragmatistic theory of knowledge. Surely an argument from analogy is not its own verification. For pragmatism the truth of a hypothesis depends upon the fact that its conceptual constructions are capable of immediate verification in terms of certain facts of immediate experience. But my neighbour's inner states of mind can never become for me objects of immediate acquaintance, unless they become my states of mind and not his, precisely in so far as he and I are distinct selves.

The hypothesis that our mental lives are similar may thus be suggested by analogy or may be stated in terms of analogy; but the analogy in question is essentially unverifiable in the required terms, i.e., in terms of immediate perceptions. For my neighbour can immediately perceive only his own states, while I, in so far as I am not my neighbour, can verify only my own states. From the point of view, then, of the argument from analogy, my neighbour, in observing his own states, does not verify my hypothesis in the sense in which my hypothesis about him demands verification, namely in terms of the experience of the self who makes the hypothesis. From this point of view, the problem of the mind of my neighbour remains hopeless.

It is possible, of course, to say of the foregoing argument from analogy what is also said both by common sense and by science, on the basis of a theory of truth which is in its essence conceptual and realistic. One can, of course, assert that in actual fact the mental states of my neighbour really exist and are in a certain relation which makes it true to say that they are analogous to mine. This real relation may be asserted to be as much a fact as any other fact in the universe. If this fact of the real analogy is granted, then it may be declared that my hypothesis to the effect that my neighbour's mind is a reality is actually true. This, however, is precisely the type of truth which William James's pragmatism undertakes to reject.

A very different appearance is assumed by the whole matter if we recognize that there is a third kind of knowledge, which is neither conceptual nor perceptual, and which is also not the sort of union of conception and perception which is completely expressible in terms of the favorite metaphor of Bergson and the pragmatists, namely, the metaphor of the conversion of conceptual credits or bank-notes into perceptual cash, i.e., into immediate data of experience. For interpretations are never verified merely through immediate data, nor through the analysis of conceptions. This is true whether I myself am the object of my own interpretation or my neighbour is in question. If we seek for metaphors, the metaphor of the conversation, already used, furnishes the best means of indicating wherein consists the relative, but never immediate, verifiability of the truth of an interpretation.

When I interpret (whether my own purposes or intents or the ideas of another man are the objects which I seek to interpret), what I first meet in experience is neither a matter of acquaintance nor a mere 'knowledge about.' What I meet is the fact that, in so far as I now understand or interpret what I call myself, I have also become aware, not immediately but in the temporal process of my mental life, that ideas have come to me which are not now my own, and which need further expression and interpretation, but which are already partially expressed through signs. Under these circumstances, what happens is that, as interpreter of these signs, I offer a further expression of what to me they seem to mean, and I make the further hypothesis that this expression makes more manifest to me both the meaning of this sign and the idea of the mind or self whereof this sign gave partial expression. It is of the essence of an expression which undertakes to interpret a sign that it occurs because the sign already expresses a meaning which is not just at the present moment our own, and which, therefore, needs for us some interpretation, while the interpretation which at the moment we offer is itself not complete, but requires further interpretation.

In literal conversation our neighbour utters words which already express to us ideas. These ideas so contrast with our own present ideas that, while we find the new ideas intelligible, and, therefore, view them as expressions of a mind, we do not fully know what they mean. Hence, in general, our neighbour having addressed us, we in reply ask him, more or less incidentally or persistently, whether or not this is what he means--i.e., we give him back our interpretation of his meaning, in order to see whether this interpretation elicits a new expression which is in substantial agreement with the expression which we expected from him. Our method in a conversation is, therefore, unquestionably the method of a 'working hypothesis.' But since this 'working hypothesis' refers to our neighbour's state of mind, it is never conceivably capable of direct verification.

Nor does what the pragmatists are accustomed to call the successful 'working' of this hypothesis consist in the discovery of any perceptible fact with which we get into merely immediate relation. Our interpretation of our neighbour satisfies our demands, precisely in so far as our interpretations which are never complete, and which always call for new expressions and for futher interpretations, lead to a conversation which remains, as a whole, essentially 'coherent,' despite its endless novelties and unexpected incidents.

Our whole knowledge of mind, in so far as by this term we mean intelligent mind, not only depends upon, but consists in, this experience of a consistent series of interpretations, which we obtain, not merely by turning conceptual 'credits' into the 'cash of immediate acquaintance,' but by seeking and finding endlessly new series of ideas, endlessly new experiences and interpretations. This never-ended series of ideas, in so far as we can hold them before our minds, tends to constitute a connected, a reasonable, a comprehensible system of ideal activities and meanings. The essence of mental intercourse--we may at once say the essence of intelligent mental life and of all spiritual relations--

not only depends upon, but consists in, this coherent process of interpretation.

Or, again, an interpretation is not a conceptual hypothesis which can be convered into 'perceptual knowledge'; it is a hypothesis which leads us to anticipate further interpretations, further expressions of ideas, novel bits of information, further ideas not our own, which shall simply stand in a coherent connexion with one another and with what the original interpretation, as a hypothesis, had led us to expect. When I deal with inanimate nature, I may anticipate facts of perception, and then my hypothesis about these facts 'work,' in so far as the expected perceptions come to pass. But, when I deal with another mind, I do not merely expect to get definable perceptions from that mind; I expect that mind to give me new ideas, new meanings, new plans, which by contrast are known at each new stage of social experience to be not my own, and which may be opposed to my own and in many respects repellent to me.

But it is essential to the social intercourse between minds that these endlessly novel ideas and meanings should, through all conflicts and novelties and surprises, retain genuine coherence. Thus, in dealing with other minds, I am constantly enlarging my own mind by getting new interpretations, both of myself and of my neighbour's life. The contrasts, suprises, conflicts, and puzzles which these new ideas present to me show me that in dealing with them I am dealing with what in some respects is not my own mind. The coherence of the whole system of interpretations, ideas, plans, and purposes shows me just as positively that I am dealing with a mind, i.e., with something which through these expressions constantly interprets itself, while, as I deal with it, I in turn constantly interpret it, and even in and through this very process interpret myself. It will and must be observed that this Alter, with which I have to deal, both in reflecting on my own mind and in seeking for new light from my neighbour, is never a merely single or separable or merely detached or isolated individual, but is always a being which is of the nature of a community, a 'many in one' and a 'one in many.' A mind knowable through interpretation is never merely a 'monad,' a single detached self; its unity, in so far as it possesses genuine and coherent unity, tends, in the most significant cases, to become essentially such as the unity which the apostle Paul attributes to the ideal Church: many members, but one body; many gifts, but one spirit (Ro 12 4ff.)--an essentially social unity, never to be adequately conceived or felt, but properly the object of what the Apostle viewed, in its practical and religious aspect, as the spiritual gift of charity, in its cognitive aspect as interpretation: pray rather that ye may interpret (1 Co 14 13).

5. Metaphysical theories of the nature of mind.--(a) Predominantly perceptual theories--The nature of mind may be defined by a given metaphysical theory mainly in terms which regard mind as best or most known through possible 'perceptions' or through possible 'acquaintance' with its nature. Such theories have been prominent throughout the whole history of human thought. They depend, first, upon ignoring the fact that what is most essential in the mind is known through the cognitive process of interpretation. They depend, further, upon making comparatively light of the effort to give any abstract conceptual description of what constitutes the essence of mind. They depend upon turning to what is sometimes called 'introspection,' or again, 'intuition,' to bring about an immediate acquaintance with mind.

Since, in general, any one who forms a predominantly perceptual idea of what mind is very naturally is not depending solely upon his own personal experience, but upon the experiences which he supposes other minds to possess, these perceptual theories of the nature of mind actually make a wide use of the reports of other people and so, more or less consciously, of arguments from analogy.

The simplest and vauguest, but in some respects the most persistent, of all theories of mental life appears, upon a largely perceptual basis, and also upon a basis of an argument from analogy, in countless forms of so-called 'animism.' Leaving aside all the historical complications, we may sum up the animistic theory of mind thus. We perceive, within ourselves, certain interesting processes which include many of our feelings, embody many of our interests, and characterize many of our activities. These activities, which in ourselves we more or less directly observe, are closely connected with the whole process of the life of the organism, i.e., of the body in whose fortunes each one of us is so interested. That which produces all these feelings, awakens in us all these interests, vitalizes our own body, and forms for each of us a centre of his own apparent world--this is the mind. The mind, then, strives and longs. It feels pain and pleasure. It prospers as the body prospers, and suffers as the body suffers.

Analogy shows that other people have such minds. These minds are as numerous as the organisms in question. They resemble one another and differ from one another, much as the organisms resemble and differ from each other. An extension of this analogy, on the basis of many motives, leads us to regard the world about us as containing many minds which are not connected with human bodies--at least in precisely the same way in which our minds are connected with our bodies. When the vast mass of superstitious beliefs which have made use of such analogies and such experiences can be more effectively controlled through the advances of the human intelligence, this primitive animism tends to pass over into theories of which we find some well-known examples in early Greek philosophies. These early Greek theories of mind appear, on a somewhat primitive and already philosophical level, as 'hylozoistic.' The world or at all events, the organic world, has life principles in it which vary as the organisms vary, and which are also of a nature that feeling and desire reveal to our relatively immediate 'knowledge of acquaintance' with our own minds.

The theories of mind of this type have played a great part in the life both of philosophy and of religion. As a general theory, animism has proved very persistent, and that for obvious reasons.

One of the Hindu Upanishads[1] well suggests both the origin and the logical basis--such as it is--of these theories when, in an allegory, it represents the question arising within the body as to where and what the soul most is. The question is disputed amongst the various bodily organs, each asserting itself to be the principal seat of life and also of mind. To discover which view is true, the members of the body take turns in leaving the organism. When the eyes go, blindness ensues, but life and mind continue, and so on with various other members. But when the breath starts to leave the body, all the other members together cry, 'Stay with us! You are the life, you are the soul, you are the self or Atman.' This allegory sufficiently indicates how primitive, how vague, and how stubborn is such a perceptual theory of mind when defined in terms of immediate intuition, and of a more or less pragmatic testing of various views about the physical organism.

Later in its origin, but continuing in its influence to the present day, is another perceptual theory of mind, which the later Upanishads present at length, and which, in another form, is exemplified by a notable assertion of H. Bergson in his Introduction to Metaphysics[2]--namely, that of one object at least we all have intuitive knowledge, this object being the self. The entire history of mysticism, the history also of the efforts to discover the nature of mind through introspection, can be summarized by means of these

[1] Brhada-ranyaka Upanishad, vl. i. 7-14, tr. in P. Duessen, Sechzig Upanishad's des Veda, Leipzig, 1897, p. 503

[2] Eng. tr., London, 1913

instances in the Hindu Upanishads that discover the true self through the experiment with breathing, and of the latest vision of Bergson, who defines the nature of mind, and also its contrast with body, in terms of the élan vital; for all these views emphasize, in various more or less primitive, or in more or less modern, forms, essentially the same theory of mind: the essence of the mind is to be known through immediate acquaintance. That which Schopenhauer calls the will to live, that which Bergson characterizes in the terms just mentioned, that which the shamans and medicine-men of all the more intelligent tribes have sought to know, is, in every case, mind viewed as an object of possible perception.

In the history of thought such perceptual theories of mind have become more highly developed and diversified, and have assumed other and very widely influential forms, by virtue of an insistence that we have an immediate perception of what is variously called 'mental activity,' 'the active soul,' or 'the principle of individual self-hood.' Motives which as a fact are not stable in purely perceptual terms have joined with this fondness for defining mind in perceptual terms to make emphatic the assertion that this theory of mind ought to be stated in expressly 'pluralistic' terms. It has, consequently, been freely asserted that we 'immediately know' our own self to be independent, to be distinct from all other selves, and thus to be unique. Since it is also sometimes asserted that we know, or that we 'know intuitively,' upon occasion, the fact that we can never be directly acquainted with the conditions of our neighbour's mind, such perceptual theories have given rise to the so-called problem of 'Solipsism.' For, if we know mind by perception only, and if we are sure of it only when we perceive it, and if each of us can perceive only his own mind, then what proves for any one of us that there is any mind but his own? The analogy which primitive animism so freely and so vaguely used becomes, for the critical consciousness questionable. In consequence, the problem of Solipsism has remained in modern times a sort of scandal of the philosophy of the mind.

The solution of the problem of Solipsism lies in the fact, upon which Peirce so well insisted,[1] that no one of us has any purely perceptual knowledge of his own mind. The knowledge of mind is not statable, in the case either of the self or of the neighbour, in terms of merely immediate acquaintance. If the truth of this proposition is once understood, the entire theory of mind, whether for metaphysics or for empirical psychology, is profoundly altered. Until this inadequacy of knowledge through acquaintance to meet the real end of human knowledge is fully grasped, it is impossible to define with success either the mind or the world, either the individual self or the neighbour.

(b) Predominantly conceptual theories.--As is the case with every highly developed doctrine, the conceptual form is very naturally assumed by any philosophical theory of mind which seeks for theoretical completeness. The conceptual theories of mind have been in history of two general types: (1) the purely conceptual, i.e. 'the abstractly rational' metaphysical theories; and (2) the more inductive conceptual theories based upon the more or less highly developed 'empirical psychologies' of the period in which these theories have flourished. We need not enumerate these theories or give their history.

Of principal importance in their history have been (1) that type of vitalism whose most classical representative is the Aristotelian theory of mind; (2) the monistic theory of mind, which often depends not so much upon the general metaphysical tendency to define the whole universe as One, but rather upon the effort to conceive mind and matter by regarding them both as the same in substance; and (3) the various

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[1] See Royce, Problem of Christianity, ii.

types of monadology, which are characterized by the assertion of the existence of many real and more or less completely independent minds or selves, whose nature it is either to be themselves persons or to be beings which under certain conditions can assume the form of persons.

Of those various important theories which are expressed in the predominantly conceptual form that of Aristotle is very deeply and interestingly related to primitive animism on the one hand, while, on the other hand, it looks towards that development of the idea of the distinct individual self upon which more modern forms of monadology have depended.

Whatever special forms the conceptual theories of mind may assume, the well-known problem remains: How are these conceptions of the various mental substances, or principles, or monads, which are each time in question related to the sorts of experience which the psychologists, the students of the natural history of mind, have at any stage of knowledge discovered or may yet hope to discover? From the point of view of modern pragmatism, conceptual theories of mind might be entertained as 'working hypotheses' if they led to verification in perceptual terms.

In fact, the modern physical sciences, in conceiving the nature of matter, deal with manifold problems, but use conceptual hypotheses regarding the nature of matter which are, in a large measure, subject to pragmatic tests. Molecules and atoms and, of late, various other types of conceptual physical entities, which were formerly supposed to be incapable of becoming objects of physical experience, now appear to come within the range of the experimenter's verifications. Therefore the processes of the experimental verification of physical hypotheses have, on the whole, a direct relation to the sort of knowledge upon which the pragmatists so much insist. The 'conceptual credits' of physical hypotheses are, on the whole, verifiable in terms of the 'perceptual cash' of laboratory experience. When this is not the case, there is a tendency towards such direct verification. Hence physical hypotheses, at least regarding what is generally called the phenomenal nature of matter, have generally proved to be topics for an inquiry within the strict realm of inductive science.

But it has been, in the past, the reproach of the conceptual theories about the nature of mind that no pragmatic test can be discovered by which one might learn what difference it would make to an observer of mental processes and, in particular, of his own mental processes whether minds are 'soul substances,' or Leibnizean monads, or not, or whether the introspective observer of his own sensations or feelings is or is not himself a Leibnizean monad or Aristotelian 'entelechy'; or again, whether he is essentially persistent and indestructible. Thus, from the pragmatic point of view, the majority of these conceptual hypotheses regarding the nature of mind show little sign of promising to prove more verifiable than they thus far have been. In consequence, the outcome of conceptual views regarding the real nature of mind has been, for many reasons, on the whole sceptical. In fact, the whole nature of mind cannot be adequately conceived, and could not be so conceived even if one's power to perceive mental processes were increased indefinitely, unless another type of cognitive processes were concerned in such an enlargement. For a mind is essentially a being that manifests itself through signs. The very being of signs consists in their demanding interpretation. The relations of minds are essentially social; so that a world without at least three minds in it--one to be interpreted, one the interpreter, and the third the one for whom or to whom the first is interpreted--would be a world without any real mind in it at all. This being the case, it might well be expected that a conceptual theory of mind would fail precisely as a perceptual theory fails. Such theories would fail because they do not view the cognitive process as it is and do not take account of that which is most of all needed in order even in the most rudimentary fashion to grasp the nature of an intelligent mind.

(c) Theories making use of the cognitive process of the interpretation.--Despite the inadequate development of the doctrine of interpretation thus far in the history of epistemology, there have not been lacking theories regarding the nature of mind according to which mind is an object to be known through interpretation, while its manifestations lie not merely in the fact that it possesses or controls an organism, but in the fact that, whether through or apart from an organism, it expresses its purposes to other minds, so that it not merely has or is a will, but manifests or makes comprehensible it will, and not merely lives in and through itself, as a monad or a substance, but is in essence a mode of self-expression which progressively makes itself known either to its fellows or to minds above or below its own grade.

That theories of mind which are based on such a view have existed, even from very primitive times, is manifest wherever in the history of religion a consultation of oracles, discovery of the future or of the will of the gods through divination, or, in fact, any such more or less superstitious appeals to other minds, and readings or interpretations of these appeals have taken place. Primitive belief in magic arts has apparently, on the whole, a conceptual type of formulation. Therefore magic has been called the physics of primitive man. It depends upon the view that man is subject to laws which, if he could discover them, he could use for his purposes, just as we now make use of the known laws of physics for industrial purposes. The supposed realm of magic arts is thus analogous to our present realm of industrial arts. The view of pragmatism--that primitive magic is not true merely because its hypotheses regarding how to cause rain or how to cure diseases do not 'work'-- is in this case fairly adequate to express the situation both epistemologically and metaphysically.

Moreover, as we have seen, animism, in its more primitive forms, expresses a predominantly perceptual theory of mind, and whether such a theory, either of mind or of the relations between mind and the physical world, is held in some simple form by the medicine-man of an obscure tribe or is impressively reiterated in a Hindu Upanishad, or is fascinatingly placed in the setting of a modern evolutionary theory by Bergson, makes comparatively little difference to the essential views of the philosophy of mind which are in question. But that view of the nature of mind which gained, apparently, its earliest type of expression when men first consulted, and hereupon more or less cautiously interpreted, the oracles of their gods has (as befits a theory of mind which is founded upon a fundamental cognitive process) persisted throughout the history of human thought. This way of viewing mind has, in fact, persisted in a fashion which enables us to distinguish its expressions with sufficient clearness from those which have had their origin either in the conceptions of primitive magic or in the perceptions which guided primitive animism.

From the point of view of the cognitive process of interpretation mind is, in all cases where it reaches a relatively full and explicit expression, equally definable in terms of two ideas--the idea of the self, and the idea of the community of selves. To an explicit recognition of what these two ideas involve a great part of the history of the philosophy of mind has been devoted. Both ideas have been subject to the misfortune of being too often viewed as reducible either to purely conceptual terms or to purely perceptual terms. If the self was defined in predominantly conceptual terms, it tended to degenerate into a substance, a monad, or a mere thing of some sort. Under the influence of a too abstract epistemology (such as the Kantian) the self also appeared as the 'logical ego,' or else as the 'pure subject.'

The fortunes of the idea of the community have been analogous. In religion this idea has proved one of the most inspiring of the ideas which have gradually transformed tribal cults into the two greatest religions which humanity possesses--Buddhism and Christianity. In ancient philosophy the community, viewed as the soul 'writ large,' inspired some of the most fruitful philosophical interpretations of Plato, Aristotle, and the Stoics. In the general history of civilization loyalty, which is identical with the practically effective love of communities as persons that represent mind on a level higher than that of the individual, is, like the Pauline charity (which is explicitly a love for the Church universal and for its spirit), the chief and the soul of the humanizing virtues--that virtue without which all the others are but 'sounding brass or a tinkling cymbal.' Yet, in the history of thought the idea of the community has greatly suffered, less frequently from the attempt to view it as the proper object of a direct mystical perception than from the tendency to reduce it to a purely conceptual form. As a conceptual object the 'mind of the community,' the 'corporate mind,' has tended to be thought of as an entity possibly significant in a legal or in a sociological sense, but difficult and perhaps unreal, in a metaphysical sense.

Experience shows, however, that the two ideas--the idea of the individual self and that of the community--are peculiarly adapted to interpret each the other, both to itself and to the other, when such interpretation is carried on in the spirit which the religion of Israel first made central in what undertook to be a world religion, and which the apostle Paul laid at the basis both of his philosophy of human history and of his Christology.

Modern idealism, both in the more vital and less formal expressions of Hegel's doctrine and in its recent efforts at a social interpretation of the self, of the course of human evolution, and of the problems of metaphysics, has already given a partial expression to a theory of which we tend to become clearly aware in proportion as we recognize what the cognitive process of interpretation is and how it contrasts with, and is auxiliary to, the processes of conception and perception. Only in terms of a theory of the threefold process of knowledge can we hope fully to express what is meant by that form of idealism which views the world as the 'process of spirit' and as containing its own interpretation and its own interpreter.

Editor's Note: This chapter is especially important because it was written after and supplements Royce's doctrine of interpretation as set forth in Volume II, of his *The Problem of Christianity*, a selection from which is included in the editor's *Anthology of Recent Philosophy*, New York, 1929, The Thomas Y. Crowell Company, pp. 174-187. See Professor John Edwin Smith's *Royce's Social Infinite — An Analysis of the Theory of Interpretation and Community*, New York, 1950, Liberal Arts Press, especially pp. 3-109. Professor Smith makes no reference to Royce's article "Mind," but bases his exposition on *The Problem of Christianity*, 2 volumes, New York, 1913, Macmillan Company. This is one proof that students of Royce's writings neglect his valuable articles contributed to Hastings' *Encyclopedia of Religion and Ethics*. Even Rand missed some of them when he compiled his bibliography (see p. iv of Preface above).

Chapter VIII

NEGATION

Editor's Note: See the editor's note above under *Axiom*, p. 125

Negation is a relative term which gets a definite meaning only when one can name or define of what, in a given case, something is the negation. In other words, there can be no negation purely in general or negation which has no definite corresponding object of which it is the negation. Any particular case of negation has its own determinate corresponding object.

I. Illustrations of various kinds of negation.--
(1) In the opening words of Hamlet's soliloquy, 'To be or not to be; that is the question,' 'not to be' involves a negation of 'to be'; both the expressions 'to be' and 'not to be' refer to possible modes of action. 'To be,' as Hamlet explains, includes in its meaning 'to bear the ills we have'; it names a mode of action which any man who chooses to continue his life decides to adopt; 'not to be' involves a course of action--namely, committing suicide--which is treated by Hamlet as the negation of continuing to live. The commandment, 'Thou shalt not steal,' commands a course of action which is the negation of that involved in stealing. Both in Hamlet's soliloquy and in the Ten Commandments, with their familiar 'Thou shalt not,' the negation of a possible course of action is considered or is commanded.

(2) Just as courses or plans of action may be the objects of negations, the negations being themselves possible courses of action which stand in the negative relation to their objects, i.e. to the courses of action

of which they are the negation, so propositions, judgments, or assertions may be the objects of negations, the meaning of the negation in each case being relative to its object. 'Charity seeketh not her own' is the negation of the proposition or judgment that would be expressed by omitting the word 'not' from the sentence. 'Ten is not a prime number' is a proposition which is the negation of the proposition 'Ten is a prime number.' When the object of the negation is a proposition, the proposition and its corresponding negation stand in the logical relation of contradiction.

(3) Negations may also have as their objects kinds or classes of beings, real or ideal. In the classes 'believer' and 'not-believer', and in the kinds of beings distinguished as 'rational' and 'non-rational,' the second term in each expression is a negative term whose object of negation is the first term.

(4) Lastly, the object of a negation may be a highly general type, grade, or state of being, to which definable characters belong or are attributed. Negations of this kind may, of course, be regarded as belonging to the previous class. But the importance of the problems or ideas involved in them may make it worth while to regard at least some of them as forming, for certain purposes, a type by themselves.

T. Harper, in The Metaphysics of the School (i. 322f.); maintains that 'evil is not a pure negation.' He expounds this thesis by saying that 'evil is a privation,' and by explaining what he means by privation. 'Privation,' in so far like 'negation' is a relative term, (see below). But the use of the word 'negation' by Harper, and by many other theologians and metaphysicians in cases of analogous complexity, is distinct from the usage which the negation of propositions or classes brings to our minds, so that it will be convenient to speak of such negations as forming a type by themselves.

In a well-known passage of The Imitation of Christ of Thomas à Kempis the adoring subject begins by praying that he may adore God, and love God, 'above' all created objects. He then enumerates, in an eloquent series, glories and powers, both of this world and of the next, accompanying each mention of some wonder, or sweetness, or beauty of the created world by the prefixed phrase, 'above all.' The passage culminates in the words, 'Above all that Thou art not, O my God.'

In this case God is explicitly regarded as in some sense the negation of the whole created world, and especially of all that is most wonderful and beautiful, and even good, in the noblest sense, about that world. Thus, to regard God as the negation of the finite world is a familiar and famous teaching of both practical and theoretical mystics.

A closely similar 'negative theology' is suggested by the legendary Hindu seer, Yājñavalkya, in his address to his wife, Maitreyī, when he says of the ātman or self, the absolute: 'The only word concerning the self is "Neti, Neti,", "It is not so, it is not so"'(SBE xv. (1900) 185). Yājñavalkya here asserts that his absolute can be defined only by means of negations. The negations, in this case, as in the case of The Imitation of Christ, make the absolute itself a negation of 'all that Thou are not,' i.e. of everything finite and relative. There is, of course, a decided distinction between the actual doctrine for which Yājñavalkya and à Kempis are contending; but they are both emphasizing an aspect of their doctrine which constitutes a sort of 'negative theology.' When the absolute is thus defined as a negation, the object of the negation being the finite world or the empirical facts and significance of the finite world, the negation differs, historically at least, and in some important respects both logically and metaphysically, from the ordinary negation of the logical text-books, whose object is a class or a kind of being.

2. The negative relation as a purely logical relation; the meaning of 'not.'--Despite the variety of the foregoing instances, it is plain that, in every negation, a characteristic relation is concerned, viz. that which is naturally expressed in our ordinary language by the

particles 'not' and 'no.' If a course of action is proposed or commanded, a dissenting voluntary agent may respond, 'I will not,' or simply 'No.' To respond in this way is to propose, threaten or promise an alternative course of action which is the negation of the original proposal, and which may be said to stand in the not-relation to it. In the case of a defined class or other universal, such as the class man or the relation brother, the class not-man and the relation not-brother stand in a relation to the class man and the relation brother which furnishes a new instance of the meaning of the word 'not' and of the general meaning of the negative relation.

The not-relation is one of the simplest and most fundamental relations known to the human mind. For the study of logic no more important and fruitful relation is known. And none has a wider range of exemplifications in the whole realm of the experience of the rational being. Anybody who can act voluntarily is able to do so by virtue of the fact that he can also refuse to act in a case where his will is concerned; i.e., a conscious voluntary action is possible only to a being who understands the meaning of 'not,' when some mode of action is its object. The importance of this understanding of the meaning of 'not' for the development of the will is exemplified in the life of childhood.

In one of the psychological efforts to observe and record the vocabulary of a young child who had recently begun to speak fluently it was noted that the two words which he most frequently used in the course of a day's speech were, first, the name that he happened to employ in speaking of himself and secondly, some word of the nature of 'no' or 'not,' used to express, not necessarily disobedient refusal, but objection, or unwillingness, or a preference and desire standing in some sort of negative contrast to the modes of action which the questions or the proposals of his elders or his playmates suggested. The vocabularies of individual children vary, of course, very widely, both in the words used and in the frequency with which they are used; but we cannot doubt how significant an advance is involved for the whole voluntary life of the child in his power to understand and use the expressions for 'no' and 'not.'

The nature of the not-relation may be most readily approached by considering the relation between a proposition and its contradictory. These are so related that, if either of them is true, the other is false, while, if either is false, the other is true; they are also so related that both of them are not true at the same time and in the same sense, while, with suitable definition of time and of sense, one of them must be true. The not-relation between two propositions is thus strictly mutual or symmetrical; i.e., if the proposition P is the negation of the proposition Q, the proposition Q is the negation of the proposition P, and conversely. Further, the relation is what may be called, in a terminology favored by a French logician, M. Couturat, 'bi-univocal'; i.e., a given proposition P cannot possess two negations, so that, if the proposition Q contradicts the proposition P, and the proposition X also contradicts the proposition P, Q and X are strictly and formally equivalent propositions. In the same way, a given proposition P is the negation of what is, essentially, one and the same proposition. Thus a proposition has only one negation, and is essentially the negation of only one proposition. Obviously connected with this fact is the familiar principle that the negation of the negation of a proposition is equivalent to the proposition itself; or, as it is often said, a double negation is equivalent to a simple affirmation.

Closely bound up with the foregoing is a fact which has caused, in its relation to more complex problems, a good deal of difficulty, both for philosophers and for common sense--that, from the purely logical point of view, there is no distinct class of propositions that are essentially affirmative, and thereby opposed to or to be distinguished from a class of propositions that are essentially negative. There are excellent reasons for distinguishing between affirmative and negative propositions so soon as we lay stress upon well-known empirical complexities and philosophically important union of ideas, which interest us when we are uniting the study of different propositions in some connected discourse. But apart from such complications and from the purely logical point of view, every proposition is the negation of its own negation. So far as the judgments of human subjects are concerned, whoever affirms any proposition to be true thereby contradicts the opinion of whatever opponent may deny the original assertion. It is vain, therefore, to say, 'For my part, I prefer to avoid negations and to confine myself to such positive affirmations as I can make'; it is vain to attempt to confine oneself to 'merely affirmative' thinking; for to affirm is to deny the contradictory of whatever one affirms. It would be equally vain for one, in a sceptical mood, to declare that his favorite attitude is that of negation or denial; for whoever denies any proposition affirms its contradictory, so that every denial is, in its logical meaning, an affirmation. In brief, it is essential to the whole business of thinking that propositions and the judgments which affirm or deny them go in pairs of contradictories--every proposition having its unique contradictory, of which, in turn, it is the unique contradictory. Hence, when Mephistopheles declares that he is 'der Geist, der stets verneint,' he asserts, from a logical point of view, precisely what is true of anybody who makes any assertions whatever.

In view of this indubitable logical fact, many very natural and important philosophical questions arise as to why affirmation and denial, as they occur in our actual thinking and discourse, appear to involve such strongly contrasted attitudes of mind, and why we regard those whose most noticeable or most usual attitude is that of affirmation as different in such important and practically potent ways from those whose habits and preferences emphasize or prefer negation. This is a problem which it is perfectly fair to consider on its merits. Despite the fact that every judgment is both positive and negative, we all actually do observe what makes us clearly distinguish, in some sense, affirming and denying as standing for decidedly different frames and attitudes of mind or states of knowledge. The attitude of Mephistopheles, that of denial, we all regard as different from the attitude which many of us prefer, that of affirmation.

This problem becomes still more important when we consider the philosophical types of negation exemplified in The Imitation of Christ and the assertions of Yājñavalkya. The worshipper in The Imitation of Christ adores a God whose divinity is defined in terms of a divine negation of the created world; the seer of the Hindu Upanishad tells of a self whose being consists in its negation of our finite distinctions. Such attitudes involve mainly negative types of thinking. Most of us, for comparatively good reasons, prefer a more positive or affirmative attitude in our assertions about both ethical and metaphysical matters. But, if every affirmation is ipso facto, from the logical point of view, a negation, since judgments, as well as propositions, essentially go in pairs of contradictories, how comes it that we so naturally and sharply oppose affirmative and negative thinking, regard Mephistopheles as engaged in some conceivable, although diabolical, task, and find the Hindu mystic, as we often say, 'too negative'? [1]

We may sum up by saying that the relation to its object for which the term 'negation' stands is, from a purely logical point of view, and apart from various empirical and philosophically important complications, one which is 'bi-univocal,' or, as it is also called a 'one-one relation,' and perfectly symmetrical. In the case of propositions the logical truth is that every proposition has its contradictory, while of two contradictories one must be true, the other false; and the

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[1] For a suggestion as to the solution of this problem see below, p. 200.)

contradictory of the contradictory of a given proposition P is precisely equivalent to the proposition P itself. For similar purely logical reasons negation, as applied to acts and modes of action, gives precisely analogous results: to every mode of action is opposed its contradictory mode of action. Of two contradictory modes of action, one who has the power to choose may put into execution the one mode; but he must choose one of the two, and he cannot choose both. One who has the opportunity and the power of choice may either steal or not steal. But one of the two he must choose. He cannot voluntarily refrain from both. Only the loss of his opportunity or of his power of voluntary choice can relieve him from being voluntarily in the position of one who steals or one who does not steal.

In the case of acts and modes of action the same complications arise as in the case of propositions and judgments. From the logical point of view, there are no modes of action which are essentially positive, and none which are essentially negative. If a man says in answer to the request to work in the vineyard, 'I go not,' his act is, logically speaking, both affirmative and negative. He negates the request 'Go work'; he takes the contradictory, but for that very reason also distinctly affirmative, attitude of positively refusing to work. No one capable of voluntary choice and possessed of the opportunity for action can undertake to do anything without thereby refusing or negating the plan of not undertaking to do that same thing. For the same reason, no voluntary agent can refuse an act without positively expressing the will not to do that act.

Yet, for all of us, positive and negative commands seem, under ordinary circumstances, to involve a distinctly different attitude of will and of mind. The contrast between the negative mode of commands illustrated by the Ten Commandments and the positive attitudes of the will expressed in the Sayings which tradition attributes to Christ has furnished a very frequent and important topic for both ethical and theological comment. 'Tell the children, in a persuasive way, what to do; but do not insist upon telling them what not to do, unless you are obliged to do so' is, at the present time, familiar pedagogical advice. It is fair to ask why the purely logical point of view, which inevitably regards negation as a symmetrical relation, seems to stand in such momentous contrast to what both common sense and experience, ethical as well as religious, so persistently exemplify.

In the case of logical classes the not-relation takes a form which we cannot here study in detail. In briefest summary, we may say that, when two terms are related as X and Not-X, the meaning of the terms is such that everything in the so-called universe of discourse to which one is confining his consideration is either X or Not-X, while nothing is both X and Not-X; the relation of negation here preserves, from the point of view of pure logic, its character as a symmetrical and 'bi-univocal' relation. Any term X which has a determinate range of application has one, and one only, corresponding negative term, or negation of X. In turn, the negative of the negative of X is the term X itself. From this point of view, pure logic, so far as we yet see, has no reason to recognize the existence of any terms except those which are essentially and equally both positive and negative terms. If a term X has a determinate meaning, then *ipso facto*, the term Not-X has a determinate meaning. The negative of the negative of X is once more X itself. Each of these terms is the negative of the other. Each is also a positive term in so far as it is the negation of its own negation. Yet common sense and ordinary experience sharply distinguish purely negative terms, or terms that are defined by negation, from terms that are positive. The reason for this difference between the logical point of view and that of common sense needs a little further explanation. We may close this elementary logical survey of the nature of the not-relation by mentioning the fact that, despite the baffling complications and abstractions with which this elementary study is beset, the not-relation remains one of the most momentous of all relations for the organization not only of all the exact sciences, but of all the systematic study of human experience and of all our knowledge concerning the order of the world and our own conduct (see further, the chapter ORDER below). If negation, considered in these formal aspects, seems barren and abstract we may assert--dogmatically enough at this stage of our inquiry--what more careful research would make clear in great detail: 'Without negation no order.' But order is not only 'heaven's first law'; it is that upon which science and righteousness, insight and ethics, equally depend.

3. Unsymmetrical relations associated with the not-relation: privation, affirmation, positive attitudes of will, and modes of knowledge.--In experience, in forming our plans of conduct and defining the topic of our discourse, the not-relation comes to appear, and in certain respects actually to be, unsymmetrical, so that there arises a significant distinction between positive, or constructive, and more purely negative modes of expression, of the description of objects, or of the formation of our plans of action. This is due to the fact that we very seldom consider the not-relation merely by itself. Both in experience and in action, both in our thoughts about things and in our observations of the real world, we find reasons for associating the not-relation with other relations, such, e.g., as are suggested by the manifold contrasts and differences which appear in our experience, and which interest both our thought and our will. When the not-relation is associated with other relations, so that we are dealing with an object P which is in certain respects to be treated as the negation of Q, while, at the same time P and Q have certain interesting differences to which we also attend, or are conceived by us under the limitations which are imposed upon us by the facts of life or by the interests of our minds, we are often able to say that only in a certain respect is P the negation of Q, or that P and Q are each the negation of the other with respect to, or within, a certain field, under

the limitations of a certain discourse, or from a certain point of view. At the same time P and Q may also be in other relations--which are not wholly symmetrical. It is under such conditions that we are led to make use of expressions such as that 'P is not the mere negation, but the privation, or the absence of Q,' or that the meaning of P implies that P expresses a certain need or want directed towards the object Q, which is then, precisely as the object of this need or want, in an unsymmetrical relation to P.

Further, it is very often the case that, in considering P and Q, we are actually limiting ourselves, our discourse, our plans of action, or our definitions to considerations and distinctions that arise within some limited field, or from the point of view of some special interest of our life, thoughts, or modes of classifying objects. We may be conscious of this limitation, or it may be merely tacit or ill-defined, or even unconscious. Within the limited field in which we are considering the distinction between P and Q, the relation between them may be or may appear to be the not-relation. Any one of very numerous considerations associated with this limitation of our point of view, our field of discourse, or our plans of action may involve relations between P and Q which are unsymmetrical, so that, as in the case of the instances of privation mentioned above, the relation between P and Q may be regarded as not symmetrical, and sometimes as associated with relations that involve objects distinct from both P and Q, with which P and Q stand in still further interesting relations. In such cases the not-relation, symmetrical and dyadic as it is, may be or may appear to be not the only relation with which we are actually concerned. Therefore, side by side with the not-relation, we may be obliged to note the existence of certain other relations in which P and Q also stand, relations triadic, tetradic, and, in fact, polyadic, with various degrees of complexity. Thus, by association with other relations, what is, from a certain point of view or in certain respects, to be regarded as the not-relation between P and Q comes to appear in other respects no longer symmetrical, and frequently no longer dyadic.

So complex are the situations and relations which under such circumstances may arise that we do well here to help ourselves by means of examples, beginning with comparatively simple instances, in order to show that most of the philosophical and empirical problems about the nature and function of negation are principally due to the fact that the conditions for negation seldom arise either in life or in science without being associated with the conditions which involve other relations than the not-relation. To unravel the tangle which this union of negation with other relations frequently involves is one of the most delicate and difficult problems of logical analysis. We can here give only the most elementary and general indications of the way in which this unravelling is to be attempted.

In Lewis Carroll's Hunting of the Snark the Barrister dreams that the Snark is 'defending the pig on the charge of deserting his sty.' In one of his pleas the Snark says: 'The charge of Insolvency fails, it is clear, if you grant the plea, "Never indebted."' The point of view from which 'insolvency' can naturally be regarded as the negation of solvency involves what constitutes--to borrow the well-known phrase used by de Morgan--a 'limited universe of discourse.' In this universe of discourse the distinction between solvency and insolvency arises; the classes 'solvent' and 'insolvent' appear as classes standing each in the not-relation to the other, and one who belongs to this universe is either solvent or insolvent, while he cannot be both. The relation between solvent and insolvent is, so far, a symmetrical one; each of the terms is the negative of the other; there is no reason to call either the essentially positive term, while the other is to be viewed as essentially negative.

The plea of the Snark is founded upon bringing to our consciousness, in a somewhat confused way, the fact that the universe of discourse whose beings are classified as solvent and insolvent is a universe of discourse of beings who are, or at some time have been, debtors, when these beings are considered with reference to the question whether they are, were, or will be able or unable to pay their debts when these debts are, were, or will be due. One who has never belonged to this universe of discourse, simply because he has never contracted a debt, certainly does not belong, so the Snark asserts, to the class of insolvent debtors, whatever else may be said about him, or whatever else is the class in which he ought to be placed. Of this limitation, whereby the universe of discourse of solvent and insolvent beings is characterized, we may be unconscious and therefore each of the classes 'solvent' and 'insolvent' appears to us as the negation of the other. That is why the relation is, in so far, treated as merely a not-relation.

If we become, as the Snark apparently wishes his listeners in the 'Shadowy Court' of the Barrister's dream to become, aware of what this limitation is, the classes 'solvent' and 'insolvent' appear in a somewhat different light. For, as even the Barrister becomes at least dimly aware, if the classes 'solvent' and 'insolvent' are classes of debtors, considered with reference to their power to pay their debts at maturity, they differ in a respect which involves other relations than the not-relation. A solvent differs from an insolvent debtor in that he possesses a power to pay at maturity. This power, if he is an honest man, he intends and probably expects to possess in due time. If he discovers that he no longer possesses it, he fails from inability to accomplish what he presumably wants to accomplish. His need is to be, if possible, solvent. An insolvent debtor is thus deprived of something that he needs or wants. His insolvency is therefore an instance of what has been called 'privation.' On the contrary, the solvent debtor has what, as an honest man, he intends or desires to have--the power to pay his debts. The relation between the solvent and the insolvent debtor is now no longer symmetrical. It is the relation between one who has and one who has not the object of a need or a desire.

For closely associated reasons, insolvency may conceivably be the object of what the Snark calls a 'charge.' The insolvent debtor may be haled into court, declared a bankrupt, or imprisoned as if for crime. From such perils and obligations the solvent debtor may be free. Here, again, the contrast between needs and privileges or possessions, between legal, social, or other empirical restrictions and freedom from such limitations, becomes important. The not-relation, in a universe of discourse thus limited, is no longer symmetrical. We need some other term than those of mere negation to express the relation involved. The insolvent and the solvent debtor classes are no longer each the mere negation of the other. Solvency appears as something positive, while insolvency involves want of something desirable, privation of something whose possession would constitute success.

Trivial as it is, this instance illustrates a type of relation which has its importance throughout the whole range of conduct, opinion, classification, conception, and so throughout the whole range of science, art, and human interests. Side by side with pure negation there now appears the distinction between two objects, each of which is in certain respects the negation of the other, while at the same time, the negation arises within some limited universe of discourse. Secondly, there appears the frequent, though the not universally present, fact that such limitations of the universe of discourse are or may be associated with empirical, conventional, legal, or ethical contrasts which lead us to regard one of the two negatively related objects as the positive, the required, the superior member of the two negations. In such cases, where, within a limited universe of discourse, the relation of negation is associated with a definable or empirically obvious distinction in value, dignity, or desireableness between the two objects, we speak of one of the two negatively related objects as involving, or as constituted by, the privation of the other; one of the two appears as the positive term, the other as what the elementary text-books of logic sometimes call 'the privative term' of the negation. Sometimes this privative term is called the 'merely negative term'--an expression more familiar than enlightening, which has helped to confuse both the popular and the technical discussions of negation.

In the case of the debtor relation we obviously have, in the universe of discourse which the Snark defines in the Barrister's dream, a limited universe. This is not the only reason why the condition of insolvency seems to involve privation. It is because the debtor wants to pay his share, or because the law may put him in peril if he does not do so, that the universe of discourse of the solvent and insolvent debtors comes to be not merely a world which is classified, but a world in which solvency, as something positive, is contrasted with insolvency, as something which involves privation. Cases where other relations than those which necessarily involve contrast and classification interesting to the will, or having different value according as X or Not-X is the term emphasized, lead to unsymmetrical relations between terms, each of which is the negation of the other, are easily to be found in exact sciences.

The whole numbers are classified into those which are prime and those which are not prime. In the universe of discourse of number, to say that ten is not a prime number is to assert the contradictory of the proposition that ten is a prime number. The limitation of the universe of discourse makes it possible to regard the prime numbers and the numbers which are not prime as in some respects unsymmetrically related. For the numbers which are not prime have factors, such that, in each case, the factors of a prime number are distinct from both the number and from unity. But the prime numbers have no such factors. Here, in so far as we are considering the purely logical character of the classification, the two classes 'prime numbers' and 'numbers not prime' are, within the universe of the numbers, negatively and symmetrically related. But the possession of factors is associated with so many other characters used in the theory of numbers, while the prime numbers (each of which initiates a new series of numbers, namely, its multiples, which from that prime number outward extend without end in order and in their due places, throughout the series of whole numbers) have so many of their properties due to this fact, that, from what one may call a purely ordinal point of view, the distinction between the prime numbers and those which are not prime is in many respects unsymmetrical. Nobody would speak of the character of being a prime number as a privation of the character of having factors. Yet the two classes, prime and not-prime, are not merely negations of each other, within their own universe of discourse. The limitation of the universe is associated with many ordinal characters, which the prime numbers possess and which numbers that have factors do not possess. From the point of view of these ordinal characters, the distinction in question thus becomes unsymmetrical.

Other interesting instances of unsymmetrical relations associated with and modifying the relation of negation are furnished by the distinction between 'continuous' and 'discontinuous' lines, aggregates of points, sets or series of numbers, 'rational' and 'irrational' numbers, 'chemical elements' and 'material substances' which are not chemical elements. In all these cases, within some limited universe of discourse, a classification involving a negation appears. At the same time some more or less important unsymmetrical relations are so bound up with the not-relation that we are certainly not dealing with mere negation.

The foregoing illustrations and considerations show how, in general, affirmation and positive and constructive attitudes of will and modes of knowledge are defined. In life we always deal with limited universes of discourse. Within these limited universes distinctions arise like those between solvency and insolvency, success and failure, acceptance and refusal, winning and losing. In all such cases the contrasts become unsymmetrical, and may be associated with extremely complicated situations, such as involve triadic or polyadic relations. Under these conditions, for reasons which may be mainly practical, and which may also be of great theoretical importance in more or less exact sciences, and may be bound up with the most various enterprises and incidents of life, conduct, and knowledge, we accept as an 'affirmative' attitude or assertion, or as a 'positive' deed or state of mind, one of two contrasted objects each of which is the negation of the other. Our reasons are of various sorts, some of which have had to serve in the foregoing as illustrations. In consequence, 'pure negation' can play no part in our concrete thinking and life, simply because it involves a merely symmetrical and logical relation between objects each of which is the negation of the other, and therefore is in a wholly symmetrical relation with the other, while there is no reason to declare one of the two negations to be the 'positive' and 'affirmative' member of the pair. It is in association with the other relations which life and experience most significantly present that negation becomes of concrete importance. When a man refuses to steal, society and the moral law are interested, not merely in the purely logical distinction between stealing and not-stealing, but also in what else the man does who does not steal.

4. The function of negation in thought and life.--In view of the distinctions which have now been illustrated, the main purpose of this article can best be accomplished by indicating the practical function which negation has in the business and conduct of life and in the work of science and philosophy.

This function is frequently defined by pointing out that what are generally called positive attitudes of mind, affirmative assertions, positive commands and exhortations, constructive thinking, and equally constructive conduct and decision are inseparable from negative attitudes, expressions, and opinions, and are implied in the latter, so that 'pure negation' is indeed impossible, while a positive attitude of mind is, in general, more fruitful and more advanced in its attainment of reasonableness than a prevailingly negative attitude.

What this article has attempted to add to the familiar philosophical lore which is thus summarized is (1) a somewhat clearer view of the general logical nature of the process of negation, and (2) an enumeration of some of the ways in which we have good reason for contrasting a prevailingly affirmative or positive way of thinking and conduct with a prevailing negative way, and for preferring affirmation to negation in certain regions and from certain points of view, as well as for certain specific purposes.

Usually, in giving the traditional preference to affirmation over negation, those who discuss the subject have failed to recognize that, in their purely logical character, both affirmation and negation, both positive and negative modes of definition, conception and counsel, illustrate the same fundamental, logical function. This, as a purely logical function, involves what is illustrated by the not-relation in general, by pairs of terms each of which is the negation of the other, and by pairs of contradictories, whether of propositions or modes of conduct. Since the not-relation, as purely logical, is symmetrical, it seems to involve, in its essential nature, no particular reason why one of two contradictory propositions should possess a form which is superior in its fruitfulness to the other, or why, of two terms each of which is the negation of the other, one should help us to conceptions essentially more fruitful than those which the other involves.

We have now seen that the reason why the logically symmetrical not-relation becomes unsymmetrical, and furnishes a pair of terms or propositions of which one is more fruitful, more instructive, or in general more valuable than the other, lies in the fact that, in a limited universe of discourse, one of two terms each of which is the negation of the other may have a value superior to that possessed by the other, and may, in any case, call to our attention matters which have an interest not possessed by the matters brought to our attention by the negation in question. We have also seen how both the experience which lies at the basis of our classification, or which warrants our proposition, and the interest which guides our will may lead us to emphasize these distinctions between the values of two terms, modes of action, or propositions which stand to each other in the not-relation. The result of our study is therefore that, when we are considering the general value which negation is to possess for us, either in the guidance of our conduct or in the clarifying and organizing of our information, we should explicitly take account (1) of the limitations of our universe of discourse, (2) of the values and interests which guide us when we consider or set in order our knowledge of this universe or direct our conduct in dealing with it, and (3) of the sort of experience which guides us as we take account of the various not-relations in question. Once more we may be aided in this summary by a reference to some of the illustrations which we have already used.

Some one advises us to prefer a positive or affirmative mode of guiding our conduct to a prevailingly negative mode, to consider what to do rather than what not to do, to give to children positive rather than negative counsel, not to take Mephistopheles for our model, to prefer constructive to prevailingly destructive modes of behaviour. What does such counsel practically mean? Whoever says, 'Do this,' logically speaking, counsels us not to refuse to do this, not to do the contradictory act. Thus, then, all counsel, in order to be positive, is also, in a strictly logical sense, negative; and, as we have seen, there is no such thing as purely negative counsel, as always denying, as the supposed purely Mephistophelian attitude. Nobody is purely constructive. Whoever builds the edifice destroys the original structure which existed before in the material out of which he constructs the edifice. Civilization implies a destruction of vast numbers of natural objects and processes. Whoever rears and trains the mature man destroys many of the natural tendencies and habits which, apart from training, nature would produce in the untrained child. Why, then, does one conceive of construction as something not negative? Why does one regard the affirmative attitude as something absolutely distinct from the negative attitude? Why does one prefer the positive in life, thought, and training?

The answer is, as we have seen, that we live in a limited universe of discourse, and that we wish to do so. The very conception of an absolutely unlimited universe of discourse would involve manifold logical contradictions, which are now well-known to logicians. Moreover, all that is valuable to us takes place in, and is subject to the limitation of, the universe of discourse of our present human life. Not only is this the case, but all the preciousness of life depends upon it. As experience shows us some of the limitations of this universe of discourse, it also reveals some of its values. Our ethical conceptions and distinctions give to many of these values a more rational character, but all the more reveal to us the importance of the asymmetries which our conduct both finds and prefers. We desire to live in houses. The desire has its well-known empirical foundation and also its rational ethical justification. Constructing houses is an activity which stands in contrast with the activity of destroying them, and which has a corresponding value. In a duly limited universe of discourse we can at pleasure so define the activities of building houses and of destroying them that the two modes of action stand in a symmetrical not-relation to each other. But in this limited universe of discourse the distinction in value between the two processes remains both empirically manifest and rationally justifiable. Of the resulting mutually negative modes of action one is called the 'positive' mode, the other involves that destructive treatment of houses which leaves people homeless, and which robs the world of its value. Therefore the counsel, 'Build rather than destroy,' has a perfectly definite warrant, which at once depends upon the logical symmetry of the not-relation in its own limited universe of discourse and makes clear why the one mode of action appears as a privation, a wiping out of values, while the other appears as both empirically and rationally preferable.

If Mephistopheles always denies, his denials which are practical as well as theoretical, are modes of action which have their place and value in a definitely limited universe of discourse, both social and ethical. In their simplest forms and instances they appear as a 'snubbing' of the proposals which others made, a sarcastic and cynical showing of contempt for human hopes and aspirations; they leave hearts desolate, ruin lives, and add to the sum of human horror. Under these circumstances, we can understand how every mode of action does indeed involve a destruction of something as well as a construction of something else, and how the not-relations involved are perfectly symmetrical, while we equally well understand why we prefer that hearts should not be made desolate, that lives should not be ruined, that the noblest in man should not be destroyed. The world in which we condemn Mephistopheles for his negation is indeed a limited universe of discourse, but the relation between heaven and hell in that world is not merely a symmetrical not-relation, but an asymmetrical relation--a relation of lower and higher, of the noblest to the basest, of the heights of justice and holiness to the depths of diabolism. It is important to see that the logical symmetry of the not-relation is needed as the basis of such unsymmetrical relations between good and evil, heaven and hell, salvation and perdition. Without negation none of these contrasts could be defined, none of these distinctions between the lower and the higher could come to clear consciousness at all;

hence negation is an absolutely essential function of our thought and will. Without negation there would be no clearness with regard to values, no knowledge of heaven or hell, of good or evil; hence Mephistopheles is indeed the inseparable companion of the one who is to learn what these distinctions are, and is even thereby to come into contact with what constitutes their value.

We turn for a moment to the case of the types of pedagogical advice which we have already mentioned. It is true that, if we give positive counsels to the children, we, logically speaking, inevitably give them advice which is also negative. For we cannot tell them what to do without counselling them not to do the contradictories of what we counsel. And, as the children are also more or less crudely logical, while some of them are more or less quaintly or crudely Mephistophelian, they will frequently find their own way of planning and performing the contradictory of what we counsel. But it is one thing to give them encouraging advice which awakens them by winning suggestions; it is another to play in our own way the part of Mephistopheles, by first finding out what their desires are and then explicitly snubbing them, and thus condemning them to the depths of discouragement, or inflaming their already existing disposition to rebel against our counsel. The Ten Commandments appear to make their appeal to an already more or less evil-minded, rebellious or wayward people, whom the thunders of the law are to terrify into submission. The use of the word 'not' gives to the Commandments this outward seeming, not because the relation of negation is logically unsymmetrical, and not because we can ever command without also forbidding the contradictory of our command, but because the limitations of the universe of discourse about Mt. Sinai, as well as the unsymmetrical distinctions between the thunders on the top of the mountain and the way downward to the plain where the people listen to the thunders, strongly suggest the overcoming by terror of an already-existing stubborn will.

On the other hand, the Sayings and the Sermon on the Mount give their counsels in a universe of discourse where the unsymmetrical relations between the Father and His children, between the Shepherd and the lost sheep, already inspire confidence, a tendency to harmony with one's counsellor, and a disposition to regard him as one who speaks with a peculiar and winning 'authority.' In such a world the not-relation is as definitely present as in any other logically definite world of counsel. On occasion the Sayings, the Parables, and the Sermon on the Mount make explicit both the not-relation and the limitations of the universe of discourse. But, on the whole, while the not-relation is logically just as prominent in the universe of discourse of the Parables and of the Sayings as it is in any other sharply defined universe of discourse, the particle 'not' does not play so large a part as in the Commandments, or as would be the case in negative appeals to the unwise or to the erring. The logic of the situation is identical. What one emphasizes in the mode of expression used is distinct; privation is in the background. What ought to be is made attractive; what ought not to be is more frequently left to be discovered by the enlightened doer of the will, who is expected 'to know of the doctrine' all the better, the more he has been won over 'to do the will.'

The practical moral of all such instances is that, both in our definition of the not-relations which interest us and in our whole use of negations, we should carefully consider the universe of discourse which we propose to employ as the field within which to make our logical distinctions, and also the asymmetrical distinctions of value which arise within that universe. The problem of the relation between these limitations and values and our use of negation is partly a psychological one, and partly one of limiting one's field of operations, for the sake of accomplishing to the full one's enterprise.

'In limitation alone can mastery be displayed.' Thus the problem of negation is one of limiting the field of attention and following the guide of the asymmetrical relations which appear within that field.

The case mentioned above, of the so-called 'negative theology' of the mystics, of The Imitation of Christ, and of the Hindu seer, still calls for a word. A 'purely negative' metaphysical doctrine is logically quite as impossible as any other 'purely negative' doctrine. For a metaphysical doctrine must consist of propositions; and a system of propositions essentially consists of a series of pairs of mutually contradictory propositions. If we call either of these propositions 'positive,' its contradictory 'negative' is its inseparable companion; if we call either 'negative,' its own contradiction, which then appears as an 'affirmative' proposition, is equally inseparable from it. But the Hindu seer, or the author of The Imitation of Christ, or any other teacher who uses expressions which illustrate a 'negative theology,' is actually thinking or speaking in a more or less deliberately limited universe of discourse. This universe of discourse is supposed to contain every thing possible, because it contains two beings, God and the world, the absolute and the finite. But an absolutely complete universe of discourse is logically impossible; and the mystic's universe of discourse is, in general, a very limited one--consisting of the objects of our more ordinary experience and the apparent object of the mystic experience itself. These two objects stand in a relation which is certainly nor merely the not-relation, although Yājñavalkya and à Kempis are unnecessarily fond of speaking as if this were so. The relation is unsymmetrical in this sense that, for the mystic, one of these objects, viz. God, the absolute, or the 'self,' is ineffably precious, and is defined in terms of the decidedly unsymmetrical relation 'above' or 'beyond,' and the other is defined as 'beneath' or sometimes as 'without.' The relation between this precious or perfect absolute object of the so-called 'negative theology' and the objects of ordinary experience is sometimes defined in terms of a contrast between 'created being' and 'uncreated being.' Now, whatever the relation of creation is, it is obviously viewed by those in question as unsymmetrical. The world 'emanates from,' or 'descends from,' or is 'produced by' its conceived Creator. The mystic God is therefore not merely and negatively uncreated, but He is that from which created being emanates or through whose will it is produced. The Hindu seers, pantheistic as they were, had still their own doctrine of 'emanation' and their various unsymmetrical relations.

It follows that the so-called 'negative theology' never tells us anything in terms of 'pure negation.' On the contrary, it very volubly characterizes a set of unsymmetrical distinctions of value, of preciousness, of grades of being, and of processes of emanation, which include numerous not-relations, but which depend for all their interest upon the fact that the mystic presents to us something of which he can say that it is best known 'when most I feel there is a lower and a higher.'

Perhaps this final illustration, when added to the foregoing, may serve to indicate the function of negation. In brief, the function of negation is, by means of the indispensable and fundamental not-relation, to lay a basis for an understanding of the complexities and asymmetries of the world of experience which may serve to clarify our ideas and systematize our conduct.

Chapter IX

ORDER

Editor's Note: See the editor's note above under *Axiom*, p. 125

I. Orderliness and its uses.--In dealing with sets or collections that consist of individual objects--sets of objects such as the stars in the sky, the men who are members of a social group, or the articles of furniture that are present in a given room--we may proceed in either of two ways.

(1) The first is the purely empirical way, which we follow when we note each individual object by itself, and then consider its relations to the other objects which belong to the collection. Thus we may take note of various chairs in one room, that one is near this window, another close to that door, and so on. Again, we may notice that, at a given time, one star is visible in the east, another is prominent in the west, and that the north star stands in such and such relations to stars which belong to the constellation called the Great Bear. This method of studying the objects which make up a given collection is of great importance, but, unless it is supplemented, it leaves us without a knowledge of the orderliness of the objects and of the collection which we study.

(2) The second is a way dependent upon our power to discover that the objects of the collection which we have studied are subject to such laws that, when we have observed some of the facts with regard to those objects, we can infer from the knowledge of these facts what may prove to be a multitude of other facts to which the objects of the same collection are also

subordinate. In so far as we can effectively draw such inferences, we are able to make the empirical knowledge which we first obtain, and which may be, so to speak, 'ruler over a few things,' into the source of a knowledge which also makes us 'rulers over many things.' That is, from the empirical knowledge which has for its object individual members of the collection which we are studying, we may be able to infer, through general laws known to us, a knowledge relating to other members of the same collection, and, on occasion, to a great many other such objects.

When a collection of objects has characters so subject to law that from a knowledge of some portion of the objects, their characters, and relations we are able to infer what are the characters and relations of at least some of the other objects, it has, in a highly general sense, the character of orderliness. The objects of this collection form in some sense an order, or what is also sometimes called an array. A closer examination shows that there are many different kinds of orderliness and order, some of which are much more important than others. But in the most general sense we may say that a collection of objects possesses order by virtue of the fact that, from a knowledge of what is true of some of its members we can infer in definite ways what is or will be true about the other objects of the collection, or about some portion of them. Order is important for us because, in the first place, by means of such properties belonging to collections we can and do economize the work both of our science and of our conduct in dealing with collections of objects which possess especially the more important kinds of order. Instead of dealing with all the details of a collection of objects, we deal with a portion of the facts, and then use our information to guide our behaviour in dealing with the rest, or with some portion of the objects.

The simplest instance of the value of order is furnished by the distinction between a confused and disorderly collection of men and an orderly array of individuals, such as is represented by soldiers drawn up in battle line, or by officials taking part in a public ceremony. If you look from a window upon a crowd of people in a park or in a market-place, and if they are not notably an ordered collection, you may make the general statement that the lack of order among them is exemplified by the fact that each individual is going his own way, so that, if you want to find out what he is doing or whither he is going, you must watch him for himself; his neighbour's doings may not be in any clearly observable relation to his own. What one is doing does not enable you to infer what others are doing. If, as in many a market-place or street, the prople are in various ways imitating one another, and are engaged in common activities, this very fact introduces, as far as it goes, some sort of order into the group. The ebb and flow of the crowd in the market-place or street, if subject to observable laws at all, makes possible the inference that some of those present are leaders in the movements which go on, while others are followers and imitators, that some preside, incite or address the crowd, or offer their wares for sale, while others are followers, or buyers, or are led or influenced by leaders or by the vendors of wares. So far as such knowledge permits you to make valid inferences from the observed facts regarding certain individuals to the observable or predictable facts regarding others, the crowd in question is not a disorderly assembly, or a collection devoid of what may be regarded as its own sort of order. The uninitiated observer who looks down upon the floor of a Stock Exchange finds a general appearance of disorder, or of the lack of order, in the collection of prople whom he at first observes. When he is better acquainted with the business going on, and with the way in which it is done, he is able to draw inferences with regard to some of the people and the modes of behaviour represented, while he learns to base his inferences upon what he observes about the people and the conduct that first attracted his attention. The observer gradually learns something about the laws followed by those who do business in the Stock Exchange, while, precisely as his knowledge grows, the people on the floor of the Stock Exchange appear to him more and more as an assemblage of persons having, and engaged in following, a more or less determinate order.

2. Law and order.--It will be observed that, in the sense which we here emphasize, order depends upon the presence of definable law, and varies with the laws which are in question. On the other hand, there is a difference between the lawfulness, or general subjection to law, which may belong to the real world, to our conduct, or to our thought, and that which we call 'order' for the purposes of the present discussion. By 'lawfulness' we mean a character which is generally viewed as belonging, not to individuals or collections of individuals, but to the general modes of behaviour, the general qualities, character, or relations which nature follows, which we regard as belonging to the real world, or which we discover when we contemplate the natural world, the metaphysically real world, or our world of thought or of conduct. But 'order' belongs to sets of individuals, to collections, to arrays of things, persons, deeds, or events. In other words, to use the term first prominently associated with the famous doctrine of Duns Scotus concerning the nature of individuals, order belongs to collections of 'haecceities,' to groups of individuals, or objects which are viewed as haecceities; but laws and lawfulness in general especially belong to our science, thought, and modes of behaviour. (See above, p. 142).

E.g., the planetary motions are subject to Kepler's laws, or to the Newtonian law of gravitation. But the solar system possesses, or is, an order, since there are some facts about planets moving in orbits external to the earth's orbit which can be inferred from this very fact. Thus from the fact that the orbit of Jupiter is related in a well-known way to the orbit of the earth, while the orbit of Venus lies between the orbit of the earth and sun, we can infer that, on occasion, Jupiter and Venus, as viewed from the earth, appear to be nearly opposite each other, while Jupiter and Saturn, being so related to the earth that the earth's orbit lies between each of them and the sun, cannot appear to us as occupying positions in the sky which are opposite to each other. These simple facts can be inferred from our knowledge of the way in which the orbit of the earth is related to the orbit of these other planets. But such facts and inferences relate to the haecceities, to the planets in question, and to their real or apparent relative positions as members of the order of the solar system.

In brief, a law of nature is an invariant mode of change which some process, or class of processes, exemplifies. Analogous definitions apply to laws and lawfulness wherever these are present in the ethical or the metaphysical world, or in any world, real or ideal, which is properly to be conceived as subject to invariant modes of change or behaviour. But an order is a set of haecceities, or of individuals, such that, by virtue of laws to which these haecceities or their general characters are subject, it is possible to draw the inferences exemplified above from some members of the order to other members of the same order.

The contrast between laws on the one hand and order on the other is easily seen in the ethical as well as in the natural realm. The moral law relates to principles and modes of conduct, and so explicitly to universals. The golden rule, the Kantian categorical imperative, Bentham's maxim regarding the choice of the greatest happiness, are all definitions of suppoedly invariant modes of action, ideal types of behaviour, which the moral law counsels for various classes or sorts of moral agents. On the other hand, in a court of law plaintiff and defendant, together with their counsel and the judge, are individuals constituting a determinate legal order. They constitute such an order because, from the fact that we know that somebody, A, is plaintiff, while somebody, T, is judge, and somebody else, perhaps D, is counsel for the plaintiff, we can infer certain other facts, with regard to the functions, interests, duties, purposes, or perils of other actual or possible members of the same court, occupied with the same business.

3. The whole numbers.--One of the most familiar and important instances of order with which the exact sciences are acquainted is the order of the so-called 'whole numbers.' This order is made up of the first member of the order, and then the sequence of numbers represented by the terms three, four, and so on. It consists of an ideally endless sequence of terms whose properties are such that a vast number of assertions can be made with regard to the properties of numbers. These assertions are, ideally speaking, as infinite in their multiplicity as is the series of whole numbers itself. Yet, logically speaking, all the arithmetic of whole numbers can be deduced from the following simple propositions which relate to elementary properties of the order in question:

(1) There is a relation which may exist between two whole numbers, and which is called the 'relation of next successor to.' Thus four is the next successor to three, two is the next successor to one: and, in general, if n is a whole number, the next successor to n is the whole number called $n + 1$.

(2) There is a whole number, and one only, which is not the next successor to any whole number. This, also called 'the first whole number,' may be conveniently represented by the symbol 0. The next successor to 0 is then called one; the next successor to one is called two, and so on.

(3) Given any number, n, then its next successor, $n + 1$, is thereby uniquely determined, so that, if every whole number has a next successor, every whole number also has but one next successor.

(4) Every whole number, without exception, has a next successor.

(5) If any property whatever is such that it belongs to the first whole number, and if it is such that, if it belongs to any whole number, it belongs to the next successor of that whole number, then this property belongs to all the whole numbers.

From these principles it is easy to show that the series of whole numbers thus defined possesses the property of being what is called 'infinite,' i.e., since every whole number has a next successor, there is no last whole number. In brief, the order of the whole numbers is such that it has a first member and no last, while every one of its members has a next successor, and while it is subject to the principle often called 'the law of mathematical induction--the law that permits the so-called 'reasoning from n to n + 1, and so to all,' in case of orders which have the same properties as those of the whole numbers. Orders of this kind have been called by A. N. Whitehead and Bertrand Russell 'progressions.' They are of enormous importance for all the exact sciences and for the whole progress of the human mind. It will be observed that one can exemplify the order of the whole numbers, by considering a very few, such as zero, one, two, three. When one thus becomes aware of the general laws to which the whole order is subject, one can deduce not merely countless theorems belonging to the arithmetic of the whole numbers, but countless properties exemplified by whole numbers not mentioned in the foregoing elementary example. The orderliness of the whole numbers and the properties both of the individual members and of possible groups of members thus become deducible from the principles just stated, and from whatever experience we have for knowing or for asserting that the order of the whole numbers is actually exemplified in the real or the ideal world. How important this knowledge of order may be we can realize if we remember how groups of individual objects or men can be arranged so as to correspond to some portion of the whole number series, while such an arrangement is useful in guiding conduct and reasoning in the most significant ways. The heads of a discourse, the stages of a plan of action, the officers or dignitaries of a given hierarchy or other numerically ordered array of individuals, the deeds of a life, the hours of the day, the days of the year, the watches turned out by a manufacturer, may be either arranged or labelled by a set of whole numbers. Such an arrangement is useful for the most manifold purposes, in planning, seeking or using objects, or in bringing individual human beings into cooperation.

4. Further illustrations.--There are cases in the realms of science, art, and life in which we deal very extensively with laws and lawfulness without paying attention to the orders in which these laws find their concrete exemplification. Thus, while our account of any given instance of order always involves a recognition of certain laws to which the members of the order are subject, we can have elaborate exposition of theories which deal with laws and their consequences in general terms, while largely neglecting to emphasize those orders in which the laws get many highly important and concrete illustrations. Thus the science of mechanics deals with the laws of motion under conditions very often conceived as ideal; and, in so far, that science does not tell us about the natural order of the physical world. For astronomy the order of the solar system has a certain primary interest, at least from one mode of approach. Newton's Principia dealt in considerable part with the laws of bodies subject to gravitation, and, in so far, did not lay stress upon the order of the solar system, but upon the laws of planetary motion and of the motion of bodies in general.

On the other hand, where our discussions relate to general laws and do not primarily lay stress upon the concrete orders that we find existing in the real or ideal world, then, in so far as they are exact and well reasoned, they inevitably include a more or less extended description of systems of ideal objects--conceptual embodiments, so to speak, of the laws the logical or the rational principles of which we are making use. In this sense any exposition of the laws to which the natural or the moral world is subject inevitably includes a presentation of some ideally ordered system of conceptual entities, of numbers, of possible deeds, or of other objects, whose array illustrates those laws with which we are dealing. Once more, the instance of the whole numbers serves to illustrate what happens when we reason about the laws of nature, or of the ideal or moral world. If the watchmaker labels his watches with numbers that stand for the order in which they were turned out of the factory, he constructs an ordered system of haecceities. This may be convenient for the process of finding lost watches, or of registering the purchase or the fortune of individual watches. On the other hand, if a man deals, as the arithmetician does, with the laws of whole numbers, he inevitably makes use of the ideal order of the whole numbers themselves. This order is constituted, not by the principles of the arithmetic of whole numbers cited above, but by the ideal haecceities, called the whole numbers themselves. On the other hand, every study of a system of law, as it becomes explicit, involves the definition of an orderly system of ideal haecceities, which exemplifies the laws in question. Thus the relations of law and order become more obvious and definite in our discussion. The maxim, 'Order is Heaven's first law,' gets at least one possible and fairly definite interpretation. Viewing heaven as a realm whose members are haecceities that belong to a world which our experience does not at present at all adequately cover, we, in faith, or in hope, regard these haecceities as having a certain array. This array will also exemplify justice, the true values which our human life was intended either to exemplify, or, in heaven, to attain. The distinction between the law and the order will be perfectly clear, precisely in so far as the laws are understood, and in so far as, in the heavenly world, the order will be needed. since in heaven justice will exist, not merely as a principle, but as the concrete order of the 'just made perfect.' Possibly the law of heaven may be, as St. Paul maintained, the law of charity. But the order of heaven will then be the order of the concrete individuals whose spiritual unity, with one another and with their Lord, the Apostle so eloquently characterizes.

5. Series and correlation of series.--The term 'series' has already been explained by the endless ideal series of the whole numbers; but there are many other series besides. We early become familiar with a new type of series when we study 'fractions' better named 'rational numbers.' The rational numbers--e.g., decimal fractions--form a series, in so far as we take account of the fact that two decimal fractions or other rational numbers which are equal to each other may be treated, for certain purposes, as if they were identical. Thus the fractions 1/2, 2/4, 3/6, and the decimal fractions .5, .50, .500, and so on, are all mutually equivalent. We may regard them, therefore, as all different representations of the same fractional value. If we confine our attention to those rational numbers called 'proper fractions,' i.e. those which lie between 0 and 1 in value, we may notice that the series of the proper fractions has the following character:

(1) When two proper fractions are distinct, i.e., when they do not possess equivalent values, there is a relation existing between them which is very familiar and possesses decidedly important properties. This may be called 'the relation of greater and less,' i.e. in the case supposed one of the fractions is the greater, while the other is the less of the two.

(2) The relation of greater and less is not a mutual relation; as the logicians sometimes say, it is asymmetrical. If a proper fraction, P is greater than a proper fraction Q, then Q is never greater than P, but stands to P in what we call the relation 'less than.' The relation 'less than' like the relation 'greater than,' is an asymmetrical relation. Each of these relations is the inverse of the other, and is, in a way, opposed to it in 'sense,' or in what may also be regarded, from a certain point of view, as 'direction.'

(3) If we choose any two rational fractions _r_ and _t_, which are not equal to each other, then there is always to be found in the series of rational numbers a third rational number which is distinct both from _r_ and from _t_. Let us call this third rational number _s_. Now _s_ may be, as the third member of this class, so chosen that _s_ is greater than _r_ and less than _t_. In this case we may say that '_s_ lies between _r_ and _t_ in the series of rational fractions.'

(4) If we choose to regard 0, not as one of the rational numbers, but as lying before all the rational numbers, and forming the inferior one of the two extremes between which all the proper fractions lie, while 1 is the superior extreme, then, as we can readily see, there is no proper fraction which is the least of all the proper fractions. For a perfectly analogous reason the series of rational fractions has no greatest member, since, whatever proper fraction we choose, such as .9999, we can always find a proper fraction which is greater than this chosen fraction, and which is nevertheless not equal to 1, so that it lies between the proper fraction which we just chose and 1.

(5) To sum up, the series of proper fractions possesses these properties: any two of its distinct members stand to each other either in a certain unsymmetrical relation of the first to the second or in the converse of this relation, so that of two proper fractions a determinate one is the greater, while the other is the less. Between any two rational fractions we can always find or determine a third which is greater than one of the pair and less than the other. There is no rational fraction which stands first in the series of proper fractions, and no rational number that stands last. The series of proper fractions has, in this sense, neither beginning nor end. Yet, if we choose, we can regard 0 and 1 as extremes so related to the entire series of the proper fractions that 0 precedes all of them, despite the fact that there is no first member in the series of proper fractions, while 1 follows all of them, despite the fact that there is no last member in the series.

(6) Last of all, we may mention a property of the 'greater-less' relation which is of cardinal importance for establishing and determining the characters which belong to the series of proper fractions. This property is expressed by saying that, if there are three proper fractions such that b is greater than a, while c is greater than b, then c is greater than a; i.e. the relation 'greater than' is not only asymmetrical, but is also what logicians call 'transitive'; it is a relation which passes over from pair to pair, or which follows what William James, in the closing chapter of his Principles of Psychology (London, 1901), calls 'the axiom of skipped intermediaries.'

The simple but highly abstract example of the series of proper fractions has, as we now see, characters which sharply distinguish it from the series of the whole numbers, in which there is a first although no last member. Corresponding to every member, n, there is its next successor, $n + 1$. On the contrary, the series of proper fractions has no first and no last member, while none of its members has either a next predecessor or a next successor. Yet the two series have certain notable features in common. In each there is a relation, which we may call 'the relation of successor,' whose converse may be regarded as the 'relation of predecessor.' This relation, so long as it is viewed as between two members of a series which are not of equivalent value, rank, or place, is unsymmetrical and transitive. We can say that, given two proper fractions which are not mutually equivalent, one is a successor of the other, in the same way in which we may call one of them greater than the other; and, if we choose two whole numbers, so long as they are not equivalent whole numbers, one of them is, in the whole number series, a successor of the other, while the other is a predecessor of the one. Different as the two series of whole numbers and proper fractions are, they still possess common and relational characters, which make both of them series. This may be viewed as a general characteristic of all those series which, like the points on a straight line in ordinary geometry, the events in a story or in a man's life, the members of a file of soldiers, or the positions of a heavenly body as it seems to move from a point in the eastern horizon to a point where it disappears in the western horizon, are possessed of the character of being 'open series,' i.e. series which do not return into themselves, and which possess no repetitions of a member.

Open series are of enormous importance for the whole theory of order. The events of time, so far as these are known to us, form open series. No event recurs. In like manner, any physical process which follows, more or less definitely, the course of an open line, be it straight or curved, presents the features of an open series. The movements of a man, when he walks once over a road and does not return, or cross his own tracks at any point, form an open series. All our business, all our plans of life, all that makes our life a progress or the reverse, all that gives ethical significance to a personality and its activities, are things dependent upon the character of the open series. In the light of the foregoing instances, we may now give a definition of the order of an open series.

Let there be a set of objects, S. The objects may be physical or ideal, theoretically or practically significant--points, numbers, deeds, people, or whatever you will. Let the members of S be subject to the following general law:

If we choose any two members of S, there will be a relation which in some way has already been exemplified by the relation 'greater and less.' This relation will apply uniformly to whatever pair of the members of S is taken into consideration, with this sole proviso, that, if you call it 'the relation G,' and if you consider two members p and q of G, then a determinate one of these two members of S, i.e. either the member p or the member q, will stand in this asymmetrical and transitive relation G to the other member of the pair. Since, by hypothesis, the relation *G* is asymetrical and transitive, if *p* stands in relation *G* to *q*, *q* will not stand in the relation *G* to *p*, but in the converse of this relation.

If all the members of *S* are subject to this general law, the members of *S* stand in the order of an open series, and actually constitute such a series. The two cases of the whole numbers and the proper fractions are instances of such a serial order.

In the form of a definition, this account of the order of an open series may be stated thus: by an 'open series' is meant a set, or collection, of objects, so that there exists, or is definable, some one relation, *G*, asymetrical and transitive, such that whatever pair, *p* and *q*, of the members of the set be chosen, one, and of necessity only one, of them stands in tne relation *G* to the other, while the other inevitably stands in the converse of the relation *G* to the first.

It is obvious that an open series conforms to our definition of what constitutes order. It is a set of objects. From some assertions regarding members of this set other assertions can be inferred. The series consists of individuals, while the asymmetrical and transitive relation, upon which each instance of a series depends, itself exemplifies a very general relational law. That the members of the series themselves illustrate this law makes it possible to infer from the relations of some of them certain relations belonging to others.

In the actual work of the sciences as well as in the formation, control, and use of serial orders, a large part is played by another set of relations, to which we must call attention in passing. In general we define various distinct series, if we have occasion to define any one series. Thus the series of the whole numbers is usually defined, not merely in the highly general and abstract manner just referred to, but more concretely, namely, in connexion with such a process as the counting of objects, or the numbering of watches, or, again, in connexion with the study of the laws of nature. The series of the proper fractions is both theoretically and practically used, not merely in dealing with abstract arithmetic, but in the processes of measurement. Concretely the proper fractions become useful to us when we are considering an ounce as a determinate subdivision of a pound, measurable by means of a certain proper fraction, or a foot as a determinate part of a yard. In other words, the abstract series of order, such as are exemplified by our proper fractions and our whole numbers, get their more concrete, and in general their more practical significance when they are brought into relation with other series.

Now the operation of connecting a serial order like the whole numbers with an ordinary process like the counting of individual things is a special instance of what logicians often call 'correlation of series.' A set of individual objects stand before me. I need, for various purposes, to count them, to know how many of them there are. I do this by using the series of whole numbers, treated, for the purposes of counting, as an order. I consider the concrete set of objects so that, by means of pointing, labelling, or some such process, I attach, in due order, each one of my whole numbers to the members of this collection, continuing until every one of the objects to be counted has been pointed at, or labelled, by one of my whole numbers. Then I regard the last one of the whole numbers of which I make use for this purpose as letting me know how many members the collection of objects which I have been counting contains.

When we are dealing not merely with collections which we can count, but with collections which we measure, we have frequent reason for correlating such series as those of the rational numbers with the various real quantities--with length, distance, weight, size, and so on. The operations upon which such correlations depend in many cases are of great complexity. Our present interest lies in the fact that by means of such processes we get our knowledge of the measurable facts of our natural world into order, and that we do so by correlating the observable or measurable series of lengths, distances, and other measurable objects, with our already known ideal and logically defined serial orders. By means of such correlations the ideal order of the abstract numbers--*e.g.*, of the whole numbers, of the rational numbers--comes to pervade, to dominate, or, as one may sometimes say, to infect, the at first less orderly or even apparently disordered world with which our experience has to deal. Order is thus correlated with the facts which the real world presents to our notice and which experience presents to be operated upon by our processes of counting, measuring, or otherwise applying our ideal series, such as whole numbers or rational numbers, to the objects of our experience. Through such correlation our conduct gets an orderly organization, which constitutes one of the most general and important consequences of our scientific study of the world. Instead of dealing with a world which seems one of chance facts, we discover what appears to be a world well arrayed, or at any rate capable of being controlled by serially ordered, precisely defined modes of action. The discovery of the whole number series was one of the first advances of the human mind in the exact sciences. All our discovery of order in nature, and all the orderly serial arrangement of our lives, ideals, and social order have been influenced by the whole number series, ever since we learned how to think in terms of this number series. Thus man first discovers order in the form of series of ideal objects, which are, indeed, suggested to him by the real world, but which he learns to understand through such constructive and ideally ordered activities as those which counting and measuring represent. Thus, by means of correlation, man continually introduces order into his real world, and is stimulated by whatever he finds orderly in that world to an effort to increase his own power to construct and to understand orderly series and their correlation.

6. Order in the moral and social world.--The foregoing accounts of instances of order as we find them in the regions with which our theoretical science deals illustrate the fact that, in so far as we take account of order, we not only gain a theoretical control over our knowledge of facts, but prepare ourselves for forms of practical activity which are made possible through the recognition, the definition, the production, and the control of order. The rows, the series, the array of real and ideal objects with which our science deals acquire their importance for us in close connexion with two principal facts, which result from the very nature of order.

(1) In so far as we are dealing with a collection of objects which, when taken together, constitute an order, we at every point economize the processes of our knowledge, and consequently make it a more powerful instrument for grasping the facts of nature and the connexions of the universe; for it is of the very nature of an order that, from a knowledge of part of the system which possesses it, we can infer what is true about other parts of the same order, and, upon occasion about the whole of the order. The general concept of material order, and of the correlation of series, has shown us how, wherever series are known to us and can be systematically correlated, we can constantly make use of some of our knowledge about the facts with which we deal to infer properties without which the advance of our knowledge would be greatly impeded.

It is customary to suppose that the most important concept of the exact sciences is the concept of quantity. That it is the characteristic work of the intellect to be guided by the effort to describe the world in quantitative terms--this is a thesis which has played a large part both in the theory and in the criticism of the work of the human intellect. The well-known Bergsonian criticism of the office and limitations of the intellect is founded upon a tendency to interpret the work of the exact sciences as, in large part, an effort to define nature, as well as reality in general, in prevailingly quantitative terms, so that, from this point of view, the intellect primarily measures, weighs, or otherwise quantitatively defines its task and its material. But this way of viewing the tasks of the intellect is as unjust to the logic of the exact sciences as it is unable to define the actual range which the conception order has in the guidance of our practical, and, above all, our ethical life.

The quantitative sciences are indeed of very great importance. But their importance is due to the fact that the quantities are subject to certain very interesting laws and types of order, which hold true for many other real and ideal systems besides those systems which the quantitative sciences study and which the arts of measurement make prominent. The science of mathematics is ill-defined as the science of quantity. On the other hand, what gives the quantitative sciences their mathematical importance is the fact that in the realm of quantities there are certain peculiarly interesting types of order present. But these quantitiative types of order are not the only exact types of order. Modern mathematical science is interested in a vast number of order types, and of orderly structures in general, which are in their nature not quantitative, and which can be neither defined nor studied in terms of quantitative relations. Geometry, by virtue both of its original name and of a good deal of its actual history, appears to be, upon its face, the science that deals with space measurement--e.g., with the measurement of lengths, areas, volumes, and similar objects. Bergson has been deceived by this aspect of it into calling our geometry 'a geometry of solids,' and into supposing that the pre-eminence which geometry has attained in our physical sciences, and which in consequence the concepts that depend on measurements have possessed in the development of all our philosophy, is due to the evolutionary accidents which have bound the human intellect to a dominant interest in the construction of solid bodies.

As a matter of fact, however, it is not an anti-intellectual tendency, but a profoundly logical interest in the purely orderly, and in the primarily non-quantitative aspect of things, that has come to be expressed in what is technically called 'non-metrical geometry.' Such a geometry science possesses in the branches of mathematics which are called 'projective geometry' and 'descriptive geometry.' These can be very highly developed without making any use of the idea of measurable geometrical quantities. Their source lies not in our power to measure, to weigh, and muscularly or mechanically to manipulate solids, but, as F.A. Enriques of Bologna has shown, in our sense of sight, in our power to notice the orderly alignment of points and sets of points, and the orderly intersections of systems of lines, as such intersections appear in the field of vision. This non-metrical or ordinal geometry may, therefore, be called 'visual geometry.' In fact the eye gives us a certain knowledge of order, distinct from that which we get through our muscles, or through various operations of measurement and metrical comparisons. The ordinal properties of the field of vision have an importance which the logic of science has neglected until recently. It is the eye that, despite all its illusions of perspective, has shown to man, from very early in his career, the distinction between heaven and earth, and the order of the heavenly movements themselves. In this sense the eye has played a large part in man's development of the conception of order. Furthermore, it is the purely ordinal aspect of the series of whole numbers and of rational numbers that lies at the foundation of some of the most important conceptions and theories of arithmetical science. In sum, then, the essence of the exact sciences lies in the fact that they reveal, as well as use, order, while quantity and the realm of the quantitative furnish only a special instance of order, not the only instance, and in certain respects by no means the most theoretically fruitful instance.

(2) With these considerations in mind, we shall now be able to make a transition to the types and the nature of order which have the greatest interest in the moral

world. As we have just seen, the order of the heavenly motions proved to be of great importance in giving men a conception of the kind of order that ought to prevail in a justly organized moral and social world. From the first, then, human conceptions of order have had as genuine a moral and social as a scientific and theoretical significance. The one great task of the intellect is to comprehend the orderly aspect of the real and of the ideal world. The conception of order lies, therefore, just as much at the basis of an effort to define our ideals of character and society as at the basis of arithmetic, geometry, or the quantitative sciences in general, or of those non-quantitative types of exact science which are now on their way to higher development. It is, therefore, not a matter of mere accident or of mere play on words that, if a man publishes a book called simply 'A Treatise on Order,' or 'The Doctrine of Order,' we cannot tell from the title whether it is a treatise on social problems or on preserving an orderly social order against anarchy or with studying those unsymmetrical and transitive relations, those operations and correlations upon which the theories of arithmetical, geometrical, and logical order depend. The bridge that should connect our logic and mathematics with our social theories is still unfinished. The future must and will find such a bridge. Then exactness of thinking will become consistent with the idealizing of conduct; the realm of the Platonic ideas that are to guide man in his search for wisdom will be conceived, at least in part, in terms of an order which will not be 'ageometrical'--not foreign in type to the sort of order which the geometricians, especially in the non-metrical part of their work, have long had reason to study. It only remains now to mention some ethical and social relations among human beings which are of importance in enabling us to infer from known facts about given human individuals what the duties, offices, and social rights and positions of other individuals either are or may become.

Among the moral and social reations of human beings there are a number of dyadic relations well known to us as furnishing a basis for serial order, and as being useful in both the lesser and the greater matters of social life. Thus the relation of superior and inferior in cases where authority is concerned enables us to define serial order. If _A_ commands _B_, and _B_ commands _C_, and if orders can be transmitted from pair to pair, then in general, or under more or less precisely definable conditions, the commands of _A_ may pass, as we often say, indirectly, through his subordinate _B_ to _B_'s subordinate _C_. In such cases it may be as well for _A_ to transmit his commands through _B_ to _C_ as to express his authority directly. How far such a series may extend and how many terms it may have will vary with the type of authority in question, with the range of its application, and so with the numbers of members who constitute the series. But, as far as the order goes, its essential characteristics are the same as those exemplified by a selected series of ordinal numbers, such as 3, 4, 5, 6. The usefulness of the idea of order is strictly analogous in the two cases. The significance of the series consisting of an officer and his subordinates, their subordinates, and so on, lies in the fact that, from a knowledge of some of the facts relating to the persons in question and to their authority, the relations of others of the facts can be deduced, and thus what is called an orderly mode of activity can be predetermined.

A relation decidedly different from that of authority, but of great practical importance, is that of some one who writes a letter, hands it to a messenger, who in his turn passes it over to some predetermined receiver of messages, while the process of indirect transmission is thus continued in an orderly way, until the letter reaches its destination. Such indirect but orderly transmission of messages may be as effective for purposes of communication as if the writer gave his letter to his correspondent without the use of intermediaries. Of such orderly transmission the conveyance of correspondence through the Post Office is a familiar example. What is essential to this sort of order is that, since from some facts you can, in an orderly system, deduce the existence of other facts, the whole undertaking of transmitting information, or other contents of letters, becomes definite, and, subject to the well-known fallibility of human conduct, predictable. The whole business world depends for the order of its transactions upon systems of organization which involve this serial order. Civilized man does most of his work through intermediaries. He pays a foreign creditor a debt by drawing upon his own local bank. He purchases in a distant part of the world by transmitting his orders through all sorts of indirect channels. What he needs to know in order to guide his actions reasonably is the same sort of thing as a student of non-metrical geometry has to recognize when he draws conclusions about an orderly array of points, or the arithmetician computes when he casts up sums of columns of figures; i.e., the civilized man, like the arithmetician, uses in his business, as the mathematician uses in his computations, some order system. It is an order system because a knowledge of part of the facts regarding its constitution enables us to reach a knowledge of other facts. In reaching this conclusion we use general principles. So far as these are exemplified by some system of individual men, of individual acts, and, in general, of haecceities, that system is an order system. Its order has for us the value that hereby we are able to arrange our modes of conduct and to predict their outcome.

As in the mathematical, so in the moral and social systems, that form of order called 'serial order' is especially familiar and important. But, wherever the system with which we deal enables us to compute, with greater or less probability, some of its facts from others supposed to be given, we are dealing with an order system.

In general, we may say that, since it is essential to order that we should be able to draw conclusions which to us are novel from knowledge about the relations of certain facts given, the most familiar features of an order system will be those which have been illustrated by the transitivity of the various pairs of members belonging to a given series.

We may say that, if by the symbol R (a b x y) I mean simply the assertion, 'The haecceities, a, b, x, and y, stand in some relation which I call the relation R, and if by the symbol S (c d x y) I mean the assertion 'The haecceities, c, d, x, y stand in the relation S to one another,' and if I am able to conclude that, in the system of objects of which I am speaking, the assertion is true that the haecceities, a, b, c, and d, stand to one another in the relation T, so that, using analogous symbols, I can write T (a b c d), and if general laws of this sort are true of the whole system with which I am dealing, then that system is in some sense an ordered system, although the property of the relations upon which I lay stress is a relational property that permits some sort of elimination. Were the laws of this elimination sufficiently known and sufficiently general, they would permit definite knowledge, and, on occasion, definite courses of action, which might rival in their orderliness the states of knowledge and courses of action which we have illustrated by the instances of the numbers and similar mathematical objects.

Such laws may be social. Were it the law of some social order that, if a, b, x, and y belong to the same social club in a great city, and if c, d, x, and y meet in the market-place of the city on a given day, as a fact a, b, c, and d will all bow to one another, and will all take off their hats, then that social order would be subject to a law which it might be worth while to know, and which would certainly give us a right to say that a, b, c, d, x, and y were, at any rate for the time in question, an orderly assemblage of persons. The order in question might not be of an externally peaceable sort. Thus we might suppose an assemblage of men subject to the law that, if a, b, x, and y fought side by side in the trenches, and if c, d, x, and y fought in opposed trenches, a, b, c, and d would, at the earliest opportunity, fraternize and cease fighting. This assemblage of men would be subject to a sort of order. The law characterizing this order might be stated in the form, that, in some definable class of instances, the comrades of certain opponents would, at the earliest opportunity, fraternize. However strange the law, it would have some sort of importance if it could be stated and put into application in some determinate manner.

Now in social and ethical matters, quite as much as in mathematical and natural matters, wherever there are laws which permit such eliminations, there is some sort of order in the system characterized by the presence of such laws. To conceive a world in which there is such order is to conceive what makes possible the realization of those ethical ideals most characteristic of organized communities. If an organized and orderly community either exists or is in the process of making, we can be loyal to it. For in such a community the individual can devote himself to activities whose fruit does not merely remain his own, but falls to the lot of the other haecceities with whom he is bound by relational ties. Order, therefore, or at least possible order, is the condition upon which depends the existence of anything lovable about our social system. If each acts only as an individual, the mere fact that he happens to be benevolent does not render his benevolence other than capricious. Loyal activity, on the other hand, is always orderly, since it involves acting in ways that are determined not merely by personal desires, or by the interests of other individuals, but by the relations in which one stands to those other individuals. Paying ones debts is a loyal act, as far as it goes. But it is an act which has no meaning for me unless I can recognize the relation of debtor and creditor. If I am not loyal, I say, in substance, "I will do this if I choose to do it." If I am loyal, I say, "I do this in case my relations to others in the community require me to do thus and thus."

It is possible, no doubt, to recognize relations without possessing the richer spirit of loyalty. Barren intellectualism is as possible in ethics as in our view of reality. But the essence of loyalty is that from the value of our relations to some things--e.g., to some individuals or haecceities--we are able to discover something about the value of our relations to other things. Loyalty which can draw no conclusions, which cannot reason from one's interest in certain haecceities and certain relations to some practically important inference about one's relations to other haecceities and other social ties, remains blind and dumb, a mere sentiment, like the luxuriantly sentimental altruism of a Rousseau, sending his own infant children to the foundling hospital, or of a Shelley, lyrically delighting in the sacrifice

> 'Of one who gave an enemy
> His plank, then plunged aside to die'
> (Prometheus Unbound, act i.),

while he ruthlessly abandons Harriet Westbrook to commit suicide, 'when the lamp is shattered,' and 'the light in the dust lies dead.'

It is essential to loyalty to draw conclusions, to live in a moral and social world which is, at least in some respects, conceived as orderly. In this sense the idea of order lies at the basis both of the ideal and of the life of any community in which loyalty is possible.

7. Law, order, and negation,--Order, as we have said, is closely connected with law. Law is some aspect of our real or ideal world, which permits us to draw inferences. It is fairly obvious that, when we know a law in terms at once general and exact, we are able, granted the suitable data, to draw a series of inferences; i.e., if certain premisses logically warrant a certain conclusion, then, in general, this conclusion may be made the basis of further inferences, which indirectly follow through the form of reasoning which the traditional text-book of logic call a 'sorites,' from the premisses with which we started. As, in a well-ordered commercial system which includes a series of banks or other agencies for the transmission of payments, one is permitted to pay one's debts more

simply, and in a more convenient way, by paying one banker, who transmits some negotiable paper to another banker, and so on to the end of the series, so, wherever an orderly system of computation, rational investigation, or definite inference in serial order is possible, one reaches conclusions which may be important by means of intermediate steps of reasoning, by orderly change of premisses and conclusion. In the case of the reasoning process the series may be interwoven in the most complex manner. In the exact sciences they are so interwoven. The order in that case is not merely an order of a simply serial type. The total result of the interwoven systems of series of inferences whereof the exact sciences consist is the development of a richer and richer system of order. The results of an old investigation become the basis of a new inquiry. One branch of exact science becomes interlaced and combined with another. What is characteristic of the process is that, whatever forms of synthesis appear, inference is everywhere an ally and instrument both in defining and in attaining at once the conception of order and the orderliness of the system with which one deals. In consequence it is one of the laws of the most purely theoretical sciences that, whatever special motives determine their development, they constantly tend to produce a richer wealth of orderliness in our own system of ideas. Upon each new stage of orderly conceptions new forms of order and of orderly systems are based. Where the methods of the inductive sciences enable us to recognize that these mathematically definable types of order have their corresponding systems of facts in the real world, our theories, developed by the process of inference, become more and more widely applicable to our understanding nature, so that the world seems to us more and more orderly. In so far as, at any point of our mental development, we see ways of creating facts, social orders and systems of social orders, which correspond to the ideas which we have so far organized, our moral and social worlds tend to become more orderly.

In brief, our power to infer, in the world of theory and of practice, both accompanies and where it is limited by our ignorance or lack of intelligence, in its turn limits our power to conceive ideal order and to understand the order of nature, and, finally, our power to give to our lives that orderliness which can win and hold our loyalty and render our life that of the spirit. And that is why the maxim, 'Let all things be done decently and in order,' is no mere expression of pedantry or formalism, but an ideal maxim, whose practical and religious significance finds its principal limitation in our ignorance or inability to give expression to our orderly ideals.

Order, then, is known to us through inference; i.e. the orderly is that which corresponds, in the real or the ideal world, to what we infer when we systematically draw conclusions from premisses. Therefore the understanding of the inmost nature of order logically depends upon understanding the relations on which our power to infer rests.

We may sum up with the observation that, if we had no exact idea of what inference is, we should have no exact idea of what order is, while our very idea of what inference is depends in all cases where an inference relates to classes and to general law, upon our idea of what constitutes the negative of a defined class of objects or cases. Without negation there is no inference. Without inference there is no order, in the strictly logical sense of the word. The fundamentally significant position of the idea of negation in determining and controlling our idea of the orderliness of both the ideal and the real world, of both the natural and the spiritual order, becomes, in the light of all these considerations, as momentous as it is, in our ordinary popular views of this subject, neglected. To the article NEGATION we must, therefore, refer as furnishing some account of the logical basis upon which the idea of order depends. From this point of view, in fact, negation appears as one of the most significant of all the ideas that lie at the base of all the exact sciences. By virtue of the idea of negation we are able to define processes of inference--processes which, in their abstract form, the purely mathematical sciences illustrate, and which, in their natural expression, the laws of the physical world, as known to our inductive science, exemplify. Serial order is the simplest instance of that orderly arraying of facts, inferences, and laws upon which, on the theoretical side of its work, science depends; while, as we have seen, in the practical world, the arraying, the organizing, of individual and social life constantly illustrates, justifies, and renders spiritually precious this type of connexion, which makes our lives consecutive and progressive, instead of incoherent and broken.

Relations of the general type of 'correspondence' enrich and interweave the various serial orders which nature, as well as our ideas, life as well as theory, present to our knowledge. If order is only one aspect of the spiritual world, it is an indispensable aspect. Without it life would be a chaos, and the world a bad dream. Loyalty would have no cause, and human conduct no meaning.

When logically analyzed, order turns out to be something that would be inconceivable and incomprehensible to us unless we had the idea which is expressed by the term 'negation.' Thus it is that negation, which is always also something intensely positive, not only aids us in giving order to life, and in finding order in the world, but logically determines the very essence of order.

Chapter X

ON DEFINITIONS AND DEBATES

Editor's Note: This article is reprinted by permission from the *Journal of Philosophy,* Vol. IX, Jan. 1912. The Committee of the American Philosophical Association to which Professor Royce refers was composed of Professors Frank Thilly, Arthur Lovejoy, Dickinson Miller, Evander B. McGilvary, and W. P. Montague. Professor Edward G. Spaulding was Secretary. See the *Journal of Philosophy,* Vol. VIII, (1911) pp. 701ff., and *The Philosophical Review,* Vol. XXI, (1912) pp. 152ff., 199ff., and 415ff.

The American Philosophical Association has lately devoted much attention to an earnest and most important effort to render its general discussions more unified, more profitable, and more conducive to the furtherance of agreement among students of philosophy. There is no doubt that both the Executive Committee of the Association and its "Committee on Definitions" have labored most self-sacrificingly to further this effort, so far as they could. Where the spirit shown has been so serious and so unselfish, criticism may appear ungracious. But the members of the committee have asked for criticisms. The issue involved is not as to their unquestionable sincerity and devotion, but as to the future policy of the Association, and as to the best way of securing, in the discussions at our meetings, the right sort of philosophical communion and community amongst the members. Our committees consist of valued and honored friends. But the Association itself is the "greater friend." We all wish it to find the best way of doing its work. We hope that it will long outlive our own generation. We want to initiate methods of cooperation which, as they come to be improved by experience, will continue to grow more and more effective as the years go on. To this end, we must be ready to criticize freely the first efforts to organize such methods of cooperation. I cheerfully submit to the severest scrutiny this my own effort at such criticism.

I

In the report of the Executive Committee, printed before the last meeting of the Association and used during the meeting, a brief statement leads to the announcement of the subject selected for debate. Those who were appointed to lead the debate, as we are told in this report, "decided to limit themselves to the discussion of 'The Relation of Consciousness and Object in Sense Perception.'" Nobody ought to doubt, I think, that this selection was a good one. Acting under the power conferred upon the Executive Committee by the previous meeting of the Association, the Executive Committee hereupon voted "to have the selection of debaters carry with it the appointment to the committee on definitions,"--the President of the Association acting as the fifth member of that committee. The committee in question, with the assistance of the Secretary of the Association, undertook, under the authority of the original vote of the Association, "the analysis and preparation of the problem for discussion," and "definitions of terms pertaining to" the "subject, for the use of those participating in the debate." That the "analysis," "the preparation of the subject," and the "definitions of terms," were, in the main, satisfactory to the leading debaters who had been appointed by the Executive Committee of the Association, was thus secured by the fact that the subject was prepared for discussion by a committee consisting of these debaters themselves with the assistance of the President and the Secretary. In their report, the Executive Committee, still acting, of course, under the authority of the Association, invited "members at large" to participate in the debate, by written papers, or otherwise, and, in doing so "to use, as far as possible, the definitions and divisions made by the committee."

The report of the Committee on Definitions, printed along with the Executive Committee's report just cited, begins by emphasizing the importance of the enterprise which the Association had thus, through the Executive Committee, assigned to its care. "Such an extensive attempt," it said, "at an organization of cooperative philosophical inquiry, has not hitherto been made by this Association." "The committee believes such organized and cooperative inquiry to have important possibilities for the future of philosophical study. It therefore ventures to express the hope that members will make a special effort to enter into the spirit of the undertaking, to review the recent literature of the subject, and, in their participation in the discussion, to conform, for the time being, to the general plan of procedure here suggested."

II

It would have been indeed a very ungracious task for any member to take part in the general discussion to which all members of the Association were thus invited, unless he could feel cordially willing to accept all the essential features of the "preparation" and of the "definitions" which in its report, the Committee on Definitions hereupon proceeded to set forth. Of the competency of the Committee to determine the rules of the proposed debate, so far as its own members were concerned, there could be of course no doubt. Of its authority, by virtue of the original vote of the Association, and under the conditions of its appointment, to ask members to follow its rulings with scrupulous care, in case they chose to participate in the general discussions at all, there could again be no doubt. The Executive Committee added its express request, as we have seen, to that of the Committee on Definitions; and hereby reasonably bound all who wanted to debate to do their best to confine their usage of terms and their definition of the issues to the forms prescribed by the Committee on Definitions. The experiment in cooperative philosophical inquiry thus for the first time tried, could not fairly be interfered with by any voluntary participant through an expression of his unwillingness--if he felt such unwillingness--to accept the Committee's analysis and definitions of the problem as sufficient for the purposes of the debate. The Committee defined certain terms: a, b, c, etc. It proposed certain questions for debate relating to matters defined in these terms. Such a question might take the form: "Are all the members of the class ab members of the class c?" It asked the members who took part in the debate to accept these definitions and formulations of questions as the topics of the inquiry. Nobody could meet the express wishes of the Committee, and discuss the topics which it wanted to have discussed, unless, accepting for the time the definitions proposed, he was ready to answer such questions as "Is every ab a member of the class c?" in the spirit of one who considered the question at issue important, and the issue well taken. If he thought the issues to be ill defined by the Committee, and unworthy of the sort of attention that the Committee required, he had no proper place in this particular experiment in cooperation. It was in that case his duty to leave the general debate to other members. For nobody was asked to debate in the meeting the question whether the Committee had well formulated the issues. Members were asked to cooperate under the rules laid down by a body authorized to restrict the field of inquiry for the sake of ensuring cooperation. Nobody could attempt the cooperation, unless he was willing to abide by the restrictions.

The responsibility of the Committee was of course as great as its authority. Itsaduty was--and no doubt its intention was--so to state the issues for debate that any or all of the philosophical opinions about these issues which are worth discussing, could be discussed. And of course a proper discussion of the issues could not include, at the meeting, such objections to the Committee's report as I now offer. The debater was required to follow the assigned rules of the game. He was not to discuss their value. He was to play under these rules. Hence, if his views about the issues were worth discussing at all, the Committee's formulas ought to have left him unhampered.

My present question is: How did the Committee accomplish this duty? Whose cooperation did it make possible, in case the one who cooperated was understood to accept the plan of debate as printed?

I am sorry that the somewhat elaborate "preparation" of the question set forth by the Committee will force me to make my answer to these questions tedious. But I can hardly be blamed for taking the Committee's formulas seriously, and, in consequence, analyzing them with care.

III

After a study of the possible issues, the Committee presented, as the first of its questions for debate, the following: "In cases where a real (and non-hallucinatory) object is involved, what is the relation between the real and the perceived object with respect (a) to their numerical identity at the moment of perception, (b) with respect to the possibility of the existence of the real object at other moments apart from any perception?" This question was to be understood by all who were to cooperate, as determined by the meanings assigned by the Committee to the terms "object," "perceived object," and "real object."

The definitions of these terms, as printed in the Committee's report, are as follows:

"By object in this discussion shall be meant any complex of physical qualities, whether perceived or unperceived and whether real or unreal.

By real objects is meant in this discussion such objects as are true parts of the material world.

By perceived object is meant in this discussion an object given in some particular actual perception."

It appears, from the context, and from the formulation of the question for debate quoted above, that the Committee very naturally laid some stress upon the fact that what is meant by "some particular actual perception" involved an occurrence at some "moment of time," called also "the moment of perception"; or, again, involved some determinate set or sequence of such momentary occurrences, "in some particular individuated stream of perceptions," that is, in the mind or in the experience of some person.

The Committee did not define what it meant by the adjective "given," used in the above-cited definition of "perceived object." Of course the participants in the discussion would seem to be in so far left free to understand and to use that word in any reasonable and customary fashion that is consistent with the context of the report; and it is plain that the members of the Committee were entirely unaware that by their use of this word they in the least restricted the reasonable liberty of anybody. As a fact, however, their definition of the term "perceived object," taken together with their formulation of their question, and the context in which they used the word given, involved a very serious interference with the range of the cooperation which they invited. For what is "given" in a "moment of perception," and what is not "given," and the sense in which anything can be "given at a particular moment," and the sense in which what is "given" can also be an "object"--all these matters are not topics of a merely pedantic curiosity about words. They are matters which have been lengthily, frequently, and momentously discussed, both in the controversies about percpetion and in other philosophical inquiries. Let us see how far and how profitably such questions could be discussed by any one who was ready to be guided, in the debate, by the rules laid down by the Committee.

IV

The word given has a wide range both of popular and of technical usage. Amongst its more technical meanings, three very readily occur to mind as possibly in question when the word is employed in a philosophical discussion.

In a very wide sense, which is rendered in special cases more determinate by the context, given means:

"Assumed, presupposed, agreed upon, accepted, taken as if it were known--but always *with reference* to some specific purpose, inquiry, undertaking, discussion, or plan of action." This sense is of course a very elastic one, and is often convenient, just because the context which further defines the plan or inquiry in question so easily specifies the conditions *subject to which* something is declared or agreed to be given. But, for this very reason, *given*, if used in this first sense, means *conditionally given*, subject to the agreements or presuppositions in question, and in this sense, does not mean: "present in some particular actual perception." In this wide sense of *conditionally given*, the Sherman Act is *given*, when legal controversies about certain combinations in restraint of trade are in question. And, for the purposes of the discussion, or of the present paper, the Committee's report, with its definitions, requests, statements of the issue, and so on, is itself *given*, to any one who wants to engage in the proposed discussion, or to read this paper. Any conceivable real or ideal object, principle, abstraction, fact, or fabulous invention, any portion of the universe, or the whole of it, could be *given*, in this sense, to somebody for some purpose. Yet the word given would not hereby be rendered hopelessly vague, because, each time, the context or other connections of the plan or inquiry that was to be undertaken would enable one to specify the conditions which made the object or principle, in this sense, hypothetically or conventionally *given*.

A second and also wide sense of the term *given* introduces the word into one's ontological vocabulary, and employs it as equivalent to *existent*, *actual*. God or an atom, Herbart's reals or Leibniz's monads, the events of history or the interior of the earth, anything believed by anybody to be a fact or a reality, may by that person be declared, in this sesne, to be a *given* fact in the world, or simply to be *given*. This meaning is of course specified, on occasion, by naming the place, time, or other definable region of being, in which the fact in question is asserted to be a fact. This signification of the word given is frequent in usage, but is often inconvenient, because of the danger of confusion between this and the third meaning of given--a danger which occasionally arises.

In a third sense, given means present to or in the "Experience" or "perception" or "feeling" or "state of mind" of somebody. I put in quotation marks the words and phrases that specify how or wherein the given is, in this sense, present, merely to indicate that, in any effort to specify this sense, one deals with matters which are amongst the most obvious and at the same time most problematic topics that philosophy has to consider. In order fully to explain what it is which in this sense is, for somebody, or at some time, given, that is, present or immediately known, or directly experienced, you need to face all the problems about "immediacy" and about "experience" and about the "self" and about "time" and about the relation of the relational aspect of the given to its non-relational aspect--all the problems, I say, which have most divided the philosophers. These are also the problems that have disturbed the seekers after some sort of "intuition" or of religious "faith," ever since the Hindoo seers first retired to the forests (or in other words "took to the woods") in their own vain effort to solve that most recondite of human mysteries, the mystery regarding what it is that is given in this third sense. From Yajnavalkya to Bergson this problem of the given has troubled men.

This sense of the word given is frequent in discussion. It is extremely useful in attempts at defining the various problems whose nature and variety have just been indicated. But unless one bears in mind how difficult and recondite these problems are, he is likely to employ the term given, in this third sense, rather to escape from facing the greatest issues of philosophy than to prepare the way for further reflection upon them. Of course an important part of the task of anybody who calls anything given, in this third sense, is to specify what sort of presentation it is upon which he is insisting.

Of these three senses of the word given, it seems plain, from the context, that the Committee intended some specification of the third sense to be in question. For their report uses the phrases: "at certain times present in a given individuated series of perceptions"; "given in some particular actual perception." Even if given were here supposed to be used in the second of the above-mentioned senses, this account of the "locus," i.e., of the place and time wherein something is for the purposes of the definition of a perceived object, given, would make the second sense (specified so as to apply to the case here in question) identical with some specification of the third sense. For even if the word given meant "is a fact," is "actual," the "perceived objects" of which the Committee speaks are here specified simply as "figuring" or as "present" "in some particular actual perception." That, then, is the way, or at least one way, in which those "perceived objects" are to be, just then, facts. And in this way the Committee means given to be understood.

As to the first sense, the Committee is not defining its "perceived objects" as given to the percipient in the sense in which the Sherman Act is given as the agreed presupposition of a legal controversy. Of course, I repeat, all of the Committee's definitions, topics, objects, and problems are to us members given, in our first sense of the word given, for the purpose of the proposed discussion, and as its agreed or at least supposed basis. But the "perceived objects" are said by the Committee to be given in "some particular actual perception," at one or at several moments of time, and in the individuated "stream" of some percipient's perceptions. The sense of given in the Committee's definition of perceived object is, therefore, some specification of the third of the senses above indicated. Hereby, then, the debater who can cooperate seems to be bound in advance by the Committee's report. In so far the wording and the context leave him not free to interpret the word given as he pleases.

What is the result? The committee has certainly not left the cooperating debater free as to his definition of the word object. An object, in this discussion, is a "complex of physical qualities." It is of course left to the debater to hold whatever view he holds as to what a "complex of physical qualities" actually is and involves. But this latter view will no longer be a matter of merely verbal conventions. Of course such "complexes" as "yellow, hard, and extended," or "brown, smooth, and solid," will be amongst the physical "objects" denoted by such phraseology. The debater will have his opinion as to what such "physical" "complexes" are, and as to what conditions they must meet in order to be "physical" at all. These views will no longer be reducible to definitions of terms. The debater's metaphysics or epistemology or perhaps just his opinions as a student of some physical science, will now come into play. If he is to cooperate, he must indeed accept the Committee's definition of object. But his doctrine about what makes a "complex" a "physical" complex, will concern issues no longer verbal, but most decidedly "material." Let us try to see what follows from this restriction of the meanings of object and of given, when taken together.

Suppose that some philosopher should be asked to cooperate whose views about what a "complex of physical qualities" is, and especially about what such a complex is when it is a "true part of the material world," required him to say: "Such an object, such a complex, however real it is (and also in case, in the Committee's sense, it is unreal), never is, and by its very nature never can be, for any human being, 'present in some particular individuated stream of perceptions,' at any moment of time; and (at least for a human being) never can be given in some particular actual perception." Suppose the philosopher held this view, not because he was disposed to favor or to dwell upon verbal controversies, but because this was his opinion as to a material issue, namely, as to what a physical "complex" is, and as to what in this sense is given. Suppose, namely, that he had inquired into what is or can be given at any moment, in any human perception, or to any human being. Suppose that he had considered, with such care as he could use, why we believe in any physical facts whatever, and what is the essential truth about the very nature of such facts, as we believe in them. Then his views would be his own, and would not depend upon his terminology. Nevertheless, when asked to cooperate, he would be bound to accept the Committee's definitions. Accepting them, what would this philosopher be obliged to say about the class of perceived objects as defined by the Committee (not, of course, as he himself would have preferred to define what he calls perceived objects)?

Such a philosopher could only say: "For a man of my opinions there exist no perceived objects (in the Committee's explicitly stated sense of that term), whether real or hallucinatory. For physical 'complexes of quality' are of such nature as forbids their being given, at any moment, in any human being's stream of perception. Therefore, for me, the Committee's class of 'perceived objects' is a 'zero-class' (in the sense of modern symbolic logic). It is an 'empty' class. Herein it resembles the class of 'horses that are not horses.'"

Since the problem of the present paper principally relates to the question: What part could a philosopher who held such views properly take in the debate, under the Committee's rules and definitions? I shall very properly be met, in my turn, at this point, by the counter-question: Are there any such philosophers? If so, are their views worth discussing?

V

In answer to this counter-question I may first cite the words of the Committee itself. On page 11 of its report, in enumerating the various current definitions of "consciousness," it refers to the following view:

"Consciousness is the instrumental activity of an organism with respect to a problematic or potential object. Thus the nature of consciousness is such as to imply the artificiality of the first question, and accordingly of its several answers." Such an opinion, then, exists. We all think it worthy of careful discussion.

I am far from defending this reported definition of consciousness; and I am very far from attempting to speak on behalf of the distinguished representative of this view to whom the Committee here refers. I can only say this: Were the reported view my own view of the nature of consciousness, I should be obliged to say that the "problematic or potential objects" to which my "instrumental activity" had "respect," were not the Committee's "perceived objects" at all; and also that if my "problematic objects" were what I supposed to be identical with the "complexes of physical qualities" which the Committee asked me to call "objects," then whatever was given in my "individuated stream of perceptions" would not be such an object. So that, in this case, the first question would be for me not only "artificial," but a question about a zero-class. And the Committee's second question, that about consciousness, would require me, if I also accepted the Committee's own definition of consciousness, to explain how this "instrumental activity" of my own organism was "that by virtue of which" the members of this zero-class--that is, the objects which for me would be no objects at all--were "numerically" or otherwise distinguished from something else. Hereupon I should indeed be at a loss how to discuss the Committee's second question any more usefully than the first question, unless, indeed, I in one way or another declined to accept the rulings of the Committee as to the conduct of the discussion, either by ignoring or by setting aside their definitions and requests. I should be sure that in any case the Committee had not succeeded in so stating the two questions as to make my opinions a natural part of the inquiry that they defined. I should feel myself excluded from profitable cooperation under the rules.

But this is no place to expound in detail the views of any one thinker. Let me next simply point out theses which every one will find more or less familiar and which, in various contexts, enter into known doctrines about perception. Let me point out that whoever holds these theses ought to regard the Committee's definition of a "perceived object" as the definitions of a zero-class.

Suppose, for instance, that one holds, with J. S. Mill, that a physical object, such as any "complex of physical qualities," is essentially "a permanent possibility of sensation" in case it is "a true part of the material world" at all, while, in case of hallucinatory or illusory physical objects, the object seems to be such a "permanent possibility" when it is not so. One who takes this view seriously, holds a doctrine which concerns not verbal definitions, but assertions as to what the object (in the Committee's sense of the term) actually is.

But a "permanent possibility of sensation" whatever else it is, is never any one sensation or group of sensations; nor yet is it any set of events in the individuated streams of perceptions of any human percipients. These events, the given facts of sensation, come and go. The "permanent possibility" is no one of them. But it is what, for Mill, the "complex of physical qualities" essentially is, and for Mill, if his doctrine were taken quite seriously, there would be no other physical objects to consider, whether real or hallucinatory. But to speak of a perceived object, in the Committee's sense, would be to speak of a fleeting sensory event, in "some given actual perception." That is, the Committee's "perceived objects" would be "permanent possibilities" that are not permanent or, once more, horses that are not horses.

Mill's account of the object of perception has often been accused of a false abstractness of formulation. Some have attempted to render his account more precise, or to deal with his arguments in another way, by asserting, with greater or less definiteness of phraseology, that the very being of a "complex of physical qualities" essentially consists in the truth of certain propositions. This doctrine, which, as it stands, is of course a metaphysical doctrine, has numerous representatives in modern discussion. Many, both before Mill's time and later, have been led to such an opinion, by considerations not wholly identical with those which Mill emphasized.

It is notable, furthermore, that, whenever such thinkers attempt to define their objects (that is, their "complexes of physical qualities" in the Committee's sense of object), with precision, they include amongst the propositions which define the being of the object certain universal propositions. Thus, for Mill, a bell to which a wire is duly attached is a "complex of physical qualities" whose being is partly defined by the truth of the proposition: "If I pull the wire I shall hear a ringing." Now any if-proposition is, in its logical sense, an universal proposition. And we are not here concerned with the material question whether this or that one amongst a set of such universal propositions is actually true, or again with the question: Subject to what conditions is it true? It is enough for our present purpose that, if a percipient is led to believe that the being of his object is in some respect defined by such a universal proposition, and if this proposition is not true, then his object is in this respect illusory. The being of the object is defined by the truth of propositions, some of which are universal, whether it is a real object or an unreal one.

In case, however, the truth of some universal proposition is essential to the constitution, to the very being, of a "complex of physical qualities," it is, once more, a contradiction in terms to talk of the truth of such an universal proposition as ever, or at any time, or to anybody, "given in some particular actual perception," such as any mortal ever has.

For any one who holds this view of what an object is, the Committee's definition of perceived object is, therefore, equivalent to the definition of a horse that is not a horse.

Now some who hold such views about physical objects are metaphysical realists. Some are Kantians; and one very important aspect of Kant's whole theory of the nature of the "phenomenal objects" which he so sharply distinguished from the sensory data, consisted in his identification of the very being of a physical object with the truth of propositions, some of which are, in his opinion, a priori and universal, while all of them are true propositions in a way that only the "spontaneity of the understanding" and the relation of the object to the transcendental "unity of apperception" could warrant or determine. Whatever the variations of Kant's own phraseology--variations easily explainable in the light of his own development--there should be no question that what his fully developed doctrine defines as the true Gegenstand of perception, and as the phenomenal, yet still perfectly objective actual "complex of physical qualities," is nothing whose nature permits it to be given to any human percipient, in any particular actual perception. Many Kantians have come to emphasize these aspects of the Kantian theory of what a "complex of physical qualities" essentially is. For all such, the Committee's definition of a "complex of physical qualities given in some particular actual perception" is a definition of "perceived objects" such that it requires some universal truth to be given as true in a particular actual moment of perception, and is also a definition which requires a permanent somewhat to be given as permanent in that which flits. The result is once more a zero-class. All such thinkers are, in my opinion, excluded from profitable participation in the Committee's discussion.

Finally, amongst those to whom the very being of a "complex of physical qualities" consists in the truth of certain propositions, whereof some are universal propositions, there are students of philosophy who are metaphysical idealists. Of these students I am one. My views are not here in question. But perhaps I have a right to say that all such metaphysical idealists, whatever their other varieties of opinion, get to their results by interpreting the truth of these propositions in terms which they suppose to be concrete and reasonable enough, but which do not permit them to admit that such truths as constitute the being of such a "complex" could be, at any moment of time, given in the stream of anybody's particular actual perceptions.

I submit that, for all such thinkers, the Committee's formulations of the issue depend upon the definition of a zero-class. All such are, in my opinion, excluded from profitable cooperation in the discussion as defined by the Committee.

In sum, whoever emphasizes the fact that what he means by a "complex of physical qualities" is something that perception brings to his notice, but that, once brought to his notice, is, in his opinion, essentially an object of interest, of belief, of intention, of faith, or of rational assurance, or of categorized conceptual structure, may well ask himself what place he has in the Committee's undertaking. For to him what is "given in a particular actual moment of perception" is simply not what he means by an object at all, whether he is a mystic or a pragmatist or a realist or an idealist.

VI

There are, then, such philosophers as I have defined, in general terms, by the assertion: For such philosophers the Committee's class of perceived objects is a zero-class. But just why, after all--so one may reply to me--why are such philosophers excluded from the inquiry proposed by the Committee? Why may they not take part if they please?

My answer has to be in terms familiar to every student of modern formal logic.

If a "zero-class" is to be the subject of an assertion, what predicates may with truth be asserted of that zero-class? The answer of modern formal logic of the prevailing neo-Boo lean type is well known, and, for logical purposes, is useful. A zero-class is not only subsumable, but is actually subsumed, under every class in the universe of discourse. Hence of any zero-class all universal propositions, whatever their predicates, are true. All particular propositions, however, which have the zero-class as their subject, are false. Hence the fortunes of a zero-class are easily to be foreordained. Thus the class defined by the term, a horse that is not a horse, is, indeed, by definition a zero-class. Hence it is formally correct to say: "All horses that are not horses can trot fast and play the violin at the same time." For the assertion is an universal. But this assertion, whose formal justification, and whose possible importance from certain points of view emphasized by modern logic, I need not here pause to explain, is no contribution to the arts or to the sciences that deal with the trotting-horse. It is an actually valuable formalism, which could indeed better be expressed in symbols. If I were asked to cooperate in a discussion amongst horse fanciers, and I had only such propositions as this to bring to their attention, it would be at once kinder and safer for me not to address the meeting. If they chose to discuss still other classes of horses that I considered to be zero-classes, I could at best only contribute the same logical truisms to their discussion, and so should be excluded from useful participation in their deliberations--unless indeed they asked me to say whether and why I thought these classes to be zero-classes. That indeed might become more a valuable and material issue, in whose discussion I might gladly take part. But if they formulated questions for debate that did not include this question, that in fact obviously excluded it, how could I further contribute, unless I undertook something in the form of a criticism of the limitations which they had put upon the debate?

As a fact, the Committee did not ask anybody to discuss the question whether there are any "perceived objects" of the precise type that is defined. Its use of its definitions, its somewhat elaborate formulation of the "logically possible views," its entire classification of the issues, excluded this inquiry from the recognized field for the debate.

No philosopher of the types illustrated in the foregoing discussion had any proper place in the cooperation which the Committee invited.

VII

Now, is all the foregoing mere "logic-chopping," mere "carping criticism," mere "verbalism," or what James loved to call "barren intellectualism"? I hope not. I intend to insist upon what I suppose to be a practical issue. It was the Committee that offered definitions supposed to be exact. My "carping" is intended only to be a taking of the Committee's requirements quite seriously. My "verbalism" consists in using their own words as they required. And my practical purpose is constructive. I want to indicate something, however little, about how our future discussions may best be organized if others at all agree with me.

That the whole issue is not merely verbal, but is quite material and of practical importance for the discussion, will appear, I think, if we simply leave out the terms defined, and substitute the definitions. In order to do this, let us consider where we should stand if the Committee had said: "Those who are to take part in this discussion are requested and supposed to assume: That 'complexes of physical qualities' may be, and often are, given in 'some particular actual perception,' at some time, and in such wise as to be 'present in some individuated sequence' or 'stream of perceptions,' and for some human being." This would not be a verbal, but a very material assumption.

Had the Committee said just this, we should have known that all whose metaphysical or epistemological opinions led them to hold, concerning physical objects, the views held by those whose otherwise very various doctrines I have just summarized, were expressly excluded from participation. Such an exclusion would have been a perfectly proper plan for the debaters who belonged to the Committee, if it was simply their intention to present their own views. But in that case the plan would not have included a call for the cooperation of members whose views were thus excluded. Now the Committee's definitions, and the preparation of the subject for debate, essentially involved, however unintentionally, just such an exclusion. This is the ground for criticism. I conceive that hereby the Committee doomed the discussion in advance to be unable to find place in any just fashion for some of the most important views about perception.

And now as to the practical result: The Committee inadvertently excluded people whom of course they never consciously intended to exclude. These people were no small party. Various mystics, scholastics, Kantians, idealists, modern realists, and pragmatists were among the people thus out of place in any inquiry that should be carried on under the restrictions carefully prepared by the Committee. When any such people attempted to enter the actual debate, they could do so only either apologetically or rebelliously or unprofitably or through an ignoring of the restrictions. This was not what the Committee intended; but it was what they brought to pass. This is not the best way to secure general cooperation. This, I think, is not what either the members of the Committee or any others of us desire to have done in our future general discussions, of which, as I hope, there will be many. The plan of having general discussions upon issures sharply defined and directly joined, is a plan that promises great results for the future, if only we learn from our first attempts how to carry out that plan better than at first we did.

What should the Committee have done? In order to answer this question, I need not dwell upon any of my own whims, prejudices, or tastes. The correct mode of procedure was suggested, during the actual general discussions, by one of the members of the Committee, namely, our devoted and highly esteemed Secretary himself. I can not quote his words, although I heard them with approval. In substance he said that one might well consider *that table yonder* (he did not define it in the abstract, but designated it by a perfectly acceptable gesture and wording), that "brown, smooth, solid somewhat"; and that one might then try to tell how he himself considered what he found "present to his senses" (namely, the *given*) to be related to what he supposed the table (the *object*) really to be. I hope that I fairly represent the Secretary's remark.

Well, *that* is the question about perception, in a nutshell. Let anybody tell (if he can, and so far as he can) *what* it is that he supposes to be *given* in his "stream of perceptions," when he looks at the "table" or "orange" or "inkstand" or whatever else he sees or otherwise perceives. Let him then indicate what this which is *given* leads him personally," at *that "moment of perception,"* to "believe to be there," or "to regard as real," or to view as a "true part of his material world," or, to consider as the object which, in his opinion, he just then knows or believes to be a "physical object." Let him hereupon compare the *given* as it is *given* with the *object* as he just then, in his momentary perception, takes to be real. Let him still further explain, if he can and will, how this object which, at the "moment of perception" he *takes to be real*, is related to what he, as a philosopher believes to be the *really* real, the genuine fact which lies at the basis both of his perception, and of the *given*, and of his momentary beliefs about "what is there." If the discussion is defined, upon the basis of such a beginning, in such wise as to call for still further comments upon known issues--let the disputant cooperate, if he will and can, by meeting these further issues. A discussion thus defined will indeed, as I firmly believe, actually illustrate the thesis that, for any percipient who wakes up to what he is believing and is doing, the being of the *object* of perception will either consist in or essentially involve the truth of certain propositions (some of them universal), each of which defines this or that aspect of the object. Since such truths by their nature exclude the possibility of their ever being given at any moment in "the stream of perceptions" of any human being, the object of perception will never be anything that is given in the personal experience of any one of us. Yet the correct result will not be (in my own opinion) what the Committee defines as "epistemological dualism and realism." It will be a result dependent upon one's definition of the truth of propositions. Hence, for me, this result will be a form of idealism which here does not concern my reader.

But the essential practical point is that, while a discussion thus initiated would need to be restricted by rules and definitions so as to keep all concerned close to the issue and in constant cooperation, there would now be no need and little danger of defining the issue or the rules or the cooperation so as to exclude anybody whose views are seriously represented in classic or current philosophical discussion.

Following the Secretary's admirable suggestion, I propose then, for the planning of our future discussions, a mode of procedure that in its origin goes back at least to Socrates or even to Zeno of Elea, and that, in its more exact and exacting restrictions, is well exemplified in the procedure of some modern mathematical logicians. It is this:

1. Define your problem as far as possible by designating typical examples. Socrates did this, and was a model for all of us. Even the Eleatic Zeno did it in his famous discussion of one of the most abstract of problems, and the issue as he defined it still interests us to-day. Our Secretary proposes to do this sort of thing in preparing our future discussions. I second the suggestion. The Committee's report did not exhaust this device before proceeding to the more abstract definitions that it had to provide. Hence these definitions were not all well adapted to their own end.

2. When designation by example has done its work, and when you come to the marshaling of the various possible varieties of opinion which you regard as worthy of discussion, it is of course natural to divide some universe of discourse into classes, and then to enumerate the possible views by pointing out the logically possible relations amongst these classes. But, when you do this, do not ignore those most momentous aspects of modern exact theories, namely, the "existence-theorems," or "existential postulates," and their contradictories (the assertions that declare or deny some of your defined classes to be "zero-classes"). Consider carefully, in the light both of formal logic and of the history of opinion, what alternatives regarding such assertations or denials--what questions as to whether one or another of your defined classes has members--are assertions or questions open to reasonable differences of opinion. This is a centrally important rule for every exact inquiry, and is greatly emphasized in the recent procedure of the logical theorists.

These are not all the rules that ought to be followed by a committee on definitions. But they are good rules, and practical rules. The Committee, on this occasion, did not follow them.

May our future discussions be controlled by committees on definitions! That is a wise plan. May the discussions prosper! That is a good hope. May the committees be as successful in practice as the present Committee was earnest and faithful in its intentions and in its toils. My carping words are ended.

Chapter XI

INTRODUCTORY NOTE TO ENRIQUE'S' PROBLEMS OF SCIENCE

Editor's Note: The Open Court Publishing Company published Mrs. Katharine Royce's translation of Federigo Enriques: *Problems of Science* in 1914. Professor Royce wrote this Introductory Note. It is reprinted by permission of the Open Court Publishing Company. The out-dated first paragraph is omitted.

The first edition of the Italian text of the Problemi della Scienza of Professor Enriques appeared in 1906, and had already become known to a wide circle of European students, belonging to various nationalities, at the time of the International Congress of Philosophy at Heidelberg, in the late summer of 1908. At this congress I myself met the author, and undertook to do what I could towards finding an American publisher for a translation of this book. Not long after the congress, Dr. Carus, on behalf of the Open Court Publishing Company, agreed to undertake the publication of the translation. The translator completed the first draft of the manuscript by June 1909. A certain amount of revision of some of the more technical portions of the translated text remained as that part of the work which I had myself, from the outset, agreed to undertake. Moderate in quantity as this task of revision has indeed proved, it came into conflict with a great number of academic and personal duties of my own,--duties which resulted from my previous engagements, and which could not at once be laid aside for the purpose of finishing my own little part of the task. Various new hindrances later intervened. In consequence of my own delay, the revised manuscript of this translation was first put in the publishers' charge as late as June 1912; and this American edition of the work of Enriques has since been in press. The delay has given opportunity to use the second Italian edition of the Problemi for the purpose of the revision of some passages of the translation.

Since the Heidelberg Congress of Philosophy in 1908, pragmatism, which, as many readers of current discussion will remember, formed the principal topic of the lively discussions of that session, has passed through its days of joyously youthful success; and is now no longer a novelty. Meanwhile, the new star of Bergson has glowed with increasing brilliancy from year to year. "Anti-intellectualism" has become, for the time, the prevailing mood in the more popular expositions of philosophy. Mobile minds,--minds characterized by what James called a "dramatic" temper,--have taken a leading part in controversy. Books such as the present one may seem for the moment, to such minds, out of place.

Yet precisely such moods as have been so widely represented in the general literature of popular philosophy since 1908, call for their own correction, or at all events for their own complement and supplement. What is most to be feared, at a time when discussion is so lively and when "anti-intellectualism" has gained such large and eager audiences, is not any definitive triumph of the "anti-intellectual" enthusiasms, but rather some too swift and "dramatic" reaction in the world of the ruling philosophical interests, some drastic return from the revolutionary temper of the thought of the moment to the older types of scientific orthodoxy, some renewal of the "dogmatic slumber" from which James, the Pragmatists, and Bergson, have awakened many plastic, quick-witted, but not always naturally judicial minds.

At just such a moment, a book like the present work may therefore be especially useful to thoughtful students, who love patience and clear ideas quite as much as they are fond of intuitions, of brilliancy, and of "vital impetus." The work of Professor Enriques stands somewhat above and apart from those philosophical controversies which the anti-intellectual movement has inspired; for this book was prepared and published in the original Italian before those controversies assumed their latest phase. Yet the author, already prominent in the discussions of the Heidelberg Congress of 1908, has since been President of the Philosophical Congress at Bologna in 1911. Translations of his Problemi della Scienza into French and German have widely extended his influence. His book is by far the most thorough and synthetic treatment of the problems of scientific methodology which belongs to recent years,--with the sole exception of the treatment which forms part of the first two volumes of Merz's History of Thought in the Nineteenth Century. Meanwhile, owing to their widely contrasting ranges and modes of discussion, Merz's book, (which is primarily a history of science, with a treatment of methodology obligato), and the book of Enriques, (which is explicitly a scientific methodology, with numerous references to contemporary interests and controversies):--these two books, I say, come into no sort of rivalry with each other, but supplement each other in a way which is all the more important because neither author can have known, I think, about the other's work until his own was substantially complete.

As for the relations of the book of Enriques to the recent controversies to which I have just referred, the work on the Problems of Science is thoroughly "intellectual" in its tone and temper, without being open to any of the usual objections to "intellectualism" which are now most popular among philosophical readers. The author (himself Professor of Projective and Descriptive Geometry in the University of Bologna), approaches his "Problems" with the training of the mathematician and the logician, and with the reputation which his treatise on "Projective Geometry" and his published essays on the "Foundations of Geometry" have long since won for him. Yet this book shows no tendency to magnify overmuch the office of the geometer, or the authority of the logician, or the powers of the human reason, in the interpretation of phenomena. Pragmatists will find Enriques emphasizing some of their own theses regarding what is now called the "instrumental" or the "functional" significance of thought, and of the whole scientific process. And this emphasis, as it appears in some of the most important general discussions (notably in the latter half of the chapter on Logic), is all the more interesting because (as we have just seen) this book,--especially in its earlier chapters,--antedates the most recent developments of pragmatism. Yet this relatively pragmatistic element of the book of Enriques appears in a form which is both largely original, and extremely many-sided and judicial. Enriques views the thinking-process as indeed an "adjustment" to "situations." But he lays great stress upon the tendency of science to seek unity, upon the synthetic aspect of scientific theory, upon what he calls the "association" of concepts and of scientific "representations." And this stress upon synthesis, this sense for wholeness and for unity, gives his treatment both of the values and of the limits of scientific hypotheses and theories, an original and a very notable character. In his view of the work and of the uses of natural science, Enriques stands in strong contrast to the original or Comtean type of "Positivism"; for he greatly emphasizes both the "objective aspect" and the significance of constructive scientific theories. As a methodologist, Enriques also finds a positive value in many "hypotheses" of such a type that Ostwald's well-known maxims of scientific method would condemn them in advance. Nor does Enriques agree with Mach's or with Pearson's limitation of the business of science to the simple "description" of physical phenomena.

Yet, despite this fondness and this respect for synthesis and for the "association" of various scientific concepts and "modes of representation," Enriques has as sincere an aversion to what he takes to be genuinely "metaphysical" constructions as has any positivist; as vigorous a hostility to the "transcendental" and to the "absolute" as is cultivated by any philosopher of our "Chicago School"; and as clear, if not as vehement, a respect for the relation between thought and will as is expressed by any Pragmatist.

What sets Enriques most apart from most of the thinkers,--pragmatists, positivists, relativists,--with whom one would be most likely to associate him,--or on occasion to confound him,--is a certain judicial temper, a breadth of view, a fondness for synthesis, an exactness of intellectual training, a love of the comparative study of his topic,--in brief a spirit which is as rare as it is requisite in a man who is to prove a thoroughly good methodologist. Enriques certainly does not, as a philosopher, blindly overrate the work or the powers of the intellect. On the contrary, he emphasizes the imperfection, the relativity, the tentative and inadequate character of all scientific and theoretical construction. Yet he is neither sceptic, nor anti-intellectualist. He does justice to the "instrumental" function of thought. But he is certainly no mere "instrumentalist." For the stress which he lays upon the "objective aspect" of even the most highly theoretical portions of scientific theory; and his insistence upon the tendency of science towards a genuine and irrevocable progress, not merely in its mutable and transient control of special experiences, but in its total view of nature,--these tendencies in Enriques seem to exclude any interpretation of his philosophy of science as a mere "instrumentalism." For Enriques, the "absolute" is no object for science. But what is won, in a scientific way, is won, and the whole tendency of the scientific attainment of truth is to be not a dealing with what is merely mutable, but an irreversible progress towards a survey of the unity of the real,--a grasping of real "invariants," and of wholes. These are theses that have a prominent place in the extremely careful, far-seeing, critical, and constructive methodology which constitutes this wealthy and well-wrought book.

Where so much is offered, it is hard to select what the reader should most consider. Personally I have taken very special interest in the treatment which Professor Enriques gives to the Principles of Geometry,--a topic which he has made especially his own, and which (as here discussed) will appeal not only to students of the logic of mathematics, but to psychologists interested in those aspects of the problem of space which especially concern their own work. The concluding chapter, dealing as it does with a wide range of highly technical physical problems and theories, is at once the most difficult (both for the translator and for the reader) and the most characteristic of the book. Here the synthetic tendencies of our author,--his wide outlook, his fairness of judgment, his careful comparisons, his bringing together of matters which are, for most readers, hopelessly far apart,--all tend to show what this book is,--a treatise on methodology such as we have long needed, and have here at length before us in English. May the work of the President of the last Congress of Philosophy serve to quicken as well as to nourish interest both in science and in methodology. May it aid us in treating more judiciously, more broadly, and more exactly, the current controversies concerning the office and the scope of the human intellect. And above all may it foster that spirit of unity in thoughtful research which its author has so well illustrated,--that spirit namely which tends to unite the work, nor only of various sciences, but of various nations.

Chapter XII

HYPOTHESES AND LEADING IDEAS

Editor's Note: Professor Royce contributed to *Science* N. S. Vol. XXXVIII, October 14, 1913, pp. 567-584 an article entitled "Some Relations between Philosophy and Science in the First Half of the Nineteenth Century in Germany," which was originally read at a meeting of the Pathological Club of the Harvard Medical School sometime during the year 1913. The first three sections of this article are historical, but the fourth deals with inductive scientific generalizations. This section is reprinted here by permission, and the editor has given it the title "Hypotheses and Leading Ideas."

Inductive scientific generalizations, in the logically simplest cases, depend upon what Mr. Charles Peirce has defined as the method of taking a "fair sample" of a chosen type of facts. Thus one who samples, to use Mr. Peirce's typical example, a cargo of wheat, by taking samples from various parts of the cargo, carefully selecting the samples so that they shall not tend to represent one part of the cargo only, but any part chosen at random, employs essentially the same inductive method which, as I gather from inquiry, Virchow used in reaching the main fundamental generalizations of his cellular pathology. Samples chosen from investigation from a great variety of growths show, both in the case of normal and in the case of morbid tissues, that in the observed samples there is sufficient evidence of the origin of each cell from a previous cell, and evidence too that the tissue is formed of generations of cells whose beginnings, both in the normal and in the morbid growths, lead back to parent cells of certain definable types. This outcome of observation, repeatedly confirmed by samples fairly chosen, that is, by samples chosen from various organisms, from various tissues, and chosen not merely to illustrate the theory, but to represent as well as may be all sorts of growth -- this, I say, leads to the <u>probable</u> assertion that this kind of origin of tissues is universal, and that one is dealing with a genuine law of nature. The probability of such a generalization can be tested in a more or less exact way, as Peirce has shown, by the principles of the mathematical theory of probabilities. Inductions of this type we may call statistical inductions. They presuppose nothing at the outset as to what laws are present in the world of the facts which are to be sampled. The technique of induction here consists wholly in learning, (1) how to take fair samples of the facts in question, and (2) how to observe these facts accurately and adequately. This kind of induction seems to be especially prominent in the organic sciences. Its logical theory is reducible to the general theory of probability, since fair samples, chosen at random from a collection of objects, tend to agree in their constitution with the average constitution of the whole collection.

But now, as you well know, a great deal of scientific work consists of the forming and testing of hypotheses. In such cases the inductive process is more complex. Peirce defines it first as the process of taking a fair sample from amongst the totality of those consequences which will be true if the hypothesis to be tested is true, and secondly as the process of observing how far these chosen consequences agree with experience. If a given hypothesis, in case it is true, demands, as often happens, countless consequences, you of course can not test all of these consequences, to see if every one of them is true. But you select a fair sample from amongst these consequences, and test each of these selected consequences of the hypothesis. If they agree with experience, the hypothesis is thereby rendered in some degree probable. The technique of induction now involves at least four distinct processes: (1) The choice of a good hypothesis; (2) the computation of certain consequences, all of which must be true if the hypothesis is true; (3) the choice of a fair sample of these consequences for a test; and (4) the actual test of each of these chosen consequences. So far as you make use of this method of induction, you need what is called training in the theory of your topic, that is, training in the art of deducing the consequences of a given hypothesis. This may involve computations of all degrees of complexity. You also need training in the art of taking a fair sample of consequences for your test; for a given hypothesis may involve numerous consequences that are already known, from previous experience, to be true. And such consequences furnish you with no crucial tests. In case of success, your hypothesis may become very highly probable. But induction never renders it altogether certain.

Classic instances of this method of induction exist in the physical sciences. In the organic sciences the process of testing hypotheses is frequent, but is less highly organized, and generally less exact than in the great cases that occur in the inorganic sciences. No theory of the consequences of any hypothesis in the organic sciences has ever yet reached the degree of precision attained by the kinetic theory of gases, or by the theory of gravitation.

So much for the two great inductive methods, as Peirce defines them. But now does successful scientific method wholly reduce to these two processes, viz., (1) sampling the constitution of classes of phenomena; and (2) sampling the theoretical consequences of hypotheses? Many students of the subject seem to think so. I think that the history of science shows us otherwise.

As a fact, I think that the progress of science largely depends upon still another factor, viz., upon the more or less provisional choice and use of what I have already called, in this paper, leading ideas.

A leading idea is, of course, in any given natural science, an hypothesis. But it is an hypothesis which decidedly differs from those hypotheses that you directly test by the observations and experiments of the particular research wherein you are engaged. Unlike them, it is an hypothesis that you use as a guide, or in Kant's phrase, as a regulative principle of your research, even although you do not in general intend directly to test it by your present scientific work. It is usually of too general a nature to be tested by the means at the disposal of your special investigation. Yet it does determine the direction of your labors, and may be highly momentous for you.

Such a leading idea, for instance, is the ordinary hypothesis that even in the most confused or puzzling regions of the natural world law actually reigns, and awaits the coming of the discoverer. We can not say that our science has already so fairly sampled natural phenomena as to have empirically verified this assumption, so as to give it a definite inductive probability. For as a fact, science usually pays small attention to phenomena unless there appears to be a definable prospect of reducing them to some sort of law within a reasonable time; and chaotic natural facts, if there were such, would probably be pretty stubbornly neglected by science, so far as such neglect was possible. On the other hand, the leading idea that law is to be found if you look for it long enough and carefully enough is one of the great motive powers not only of science but of civilization.

It may interest you to know that the modern study of the so-called axioms of geometry, as pursued by the mathematicians themselves, has shown that such principles as the ordinary postulate about the properties of paralled lines (as Euclid defines that postulate) are simply leading ideas. What the text-books of geometry usually assert to be true about the fundamental properties of parallel lines is a principle that is neither self-evident, nor necessarily true, nor even an inductively assured truth of experience. It turns out, in the light of modern logical mathematical analysis, to be, I say, simply a leading idea, -- that is, a principle which we can neither confirm nor refute by any experience now within our range, but which we use and need in geometry precisely because it is so serviceable in simplifying the geometry of the plane.

If I may venture to cite an example from your own science, I should suggest the following: That fundamental principle of Virchow's "Cellular Pathology" which asserted the origin of every cell from a cell was, as I already said, a perfectly straight-forward induction, of Peirce's first type, that is, it was a probable assertion of a certain constitution as holding for a whole type of cases -- an assertion made simply because this constitution had been observed to hold for a sufficient number of fairly selected samples of the type. But, on the other hand, consider another principle which Virchow asserted already in 1847 or earlier, and which, as I have long been told, has been of the first importance for the whole later development of your science: "We have learned to recognize," says Virchow, "that diseases are not autonomous organisms, that they are no entities that have entered into the body, that they are no parasites which take root in the body, but that they merely show us the course of the vital processes under altered conditions" ("das sie nur den Ablauf der Lebenserscheinungen unter veränderten Bedingungen darstellen").

Now of course I have nothing to suggest regarding the objective truth of this assertion. But I venture to point out that, logically regarded, it is not an hypothesis to be definitely tested by any observation, but is rather an hypothesis of the type of Euclid's postulate about the parallel lines, that is, it is a leading idea. For, on the one hand, how could Virchow regard this principle as one that had been definitely tested, and already confirmed by direct observation and experience at a time when, as in 1847, he was not yet possessed even of his own general principle of a cellular pathology, and when he regarded the whole science of pathology as in its infancy, and the causation of disease as very largely unknown? On the other hand, what experience could one look for that would definitely refute the principle if it were false? Would the experience of such facts as those of your modern bacteriology refute that principle? No, at least so far as I understand the sense of the principle as Virchow stated it in 1847. For when bacteria, or when any of their products or accompaniments came to be recognized either as causing disease, or as affecting the course of disease in any way, it was still open to Virchow to say that the causes thus defined simply constitute these very veränderte Bedingungen under which the Ablauf der Lebenserscheinungen takes place. In other words, the principle, if understood with sufficient generality, simply asserts that a disease can not occur in an organism without the process of the disease being themselves alterations of the processes of the organism, and such alterations as the altered conditions, whatever they are, determine. Such a principle, so understood, seems tolerably safe from empirical refutation. It would remain unrefuted, and empirically irrefutable, so far as I can see, even if the devil caused disease. For the devil would then simply be one of the veränderte Bedingungen. Thus when the devils on a famous occasion entered, in the tale, into the Gaderene swine, the Ablauf of the Lebenserscheinungen of the swine was such, under the veränderte Bedingungen, that, as we are told, they ran down a steep place into the sea. But I do not see that this just stated pathological postulate of Virchow's need have suffered shipwreck, or need even have received any damage, even on this occasion. The devils are indeed represented in the tale as entities that from without entered into the swine, as bullets might have done. But the running down into the sea is nur der Ablauf der Lebenserscheinungen of the swine themselves. Let bullets or bacteria, poisons or compressed air, be the Bedingungen, the postulate that Virchow states will remain irrefutable, if only it be interpreted to meet the case. For the principle merely says that whatever entity it may be, fire or air or bullet or poison or devil, that affects the organism, the disease is not that entity, but is the changed process of the organism. What then is this hypothesis, this rejection of every external-entity-theory of disease, as the hypothesis appears when Virchow writes these words in 1847? I reply, this is no hypothesis in the stricter sense; that is, it is no trial proposition to be submitted to precise empirical tests. It is, on the contrary, a very precious leading idea. It is equivalent to a resolution to search for the concrete connection between the processes of any disease and the normal process of the organism, so as to find the true unity of the pathological and the normal process through such a search. Without some such leading idea, the cellular pathology itself could never have resulted; because the facts in question would never have been observed. And I suppose that some equivalent leading idea, if not precisely that which Virchow stated in 1847, is just as precious to you to-day in your own pathological work.

The value of such leading ideas for a science lies in the sorts of research that they lead men to undertake, and also in the sorts of work that they discourage. They are, I repeat, regulative principles. Observation does not, at least for the time, either confirm or refute them. But, on the other hand, they awaken interest in vast ranges of observation and experiment, and sustain the patience and enthusiasm of workers through long and baffling investigations. They organize science, keep it in touch with the spirit of the age, keep alive in it the sense of the universal, and assure its service to humanity. Specialism, without leading ideas, remains but a sounding brass and a tinkling cymbal.

The sources of useful leading ideas seem to me to be various. Social, and in particular industrial interests, suggest some of them, as the perennial need of paying the coal-bills for the steam engines suggested, as we have seen, one of the leading ideas which pointed the way towards the modern theory of energy. The comparison of the results of various sciences awakens such leading ideas in various minds. Schleiden set Schwamm searching for the basis of the cell theory in animal tissues. That was the suggestion of an hypothesis in the narrower sense, to be tested. But when the physical sciences set the students of organic science to the work of conceiving organic processes as mechanical in their inmost nature, that was the suggestion of a leading idea.

But another source of such leading ideas has been, upon occasion, philosophy. Philosophy itself might be defined as a systematic scrutiny of leading ideas. It has also proved to be often an inventory and interpreter of such ideas.

Chapter XIII

INTRODUCTION TO POINCARÉ'S FOUNDATIONS OF SCIENCE

Editor's Note: In 1913 The Science Press published the first edition of the English translation by George Bruce Halsted of H. Poincaré's *The Foundations of Science*. At the request of the translator, who had been one of his fellow students, and of Professor J. McKeen Cattell, editor of Science, Professor Royce wrote a special introduction for this important treatise. The book was reprinted in 1921. By special permission of The Science Press, this informative introduction is here reproduced, except that the brief opening paragraph is omitted. Two sections of this introduction are especially worthy of note. Section II contains a clear statement of a thesis of logical empiricism, which has been voluminously elaborated since Royce wrote. Section IV gives another statement of Royce's concept of hypotheses as leading ideas, together with Virchow's theoretical principle of cellular pathology (see the end of the preceding chapter).

I

The branches of inquiry collectively known as the Philosophy of Science have undergone great changes since the appearance of Herbert Spencer's *First Principles,* that volume which a large part of the general public in this country used to regard as the representative compend of all modern wisdom relating to the foundations of scientific knowledge. The summary which M. Poincaré gives, at the outset of his own introduction to the present work, where he states the view which the 'superficial observer' takes of scientific truth, suggests, not indeed Spencer's own most characteristic theories, but something of the spirit in which many disciples of Spencer interpreting their master's formulas used to conceive the position which science occupies in dealing with experience. It was well known to them, indeed, that experience is a constant guide, and an inexhaustible source both of novel scientific results and of unsolved problems; but the fundamental Spencerian principles of science, such as 'the persistence of force,' the 'rhythm of motion' and the rest, were treated by Spencer himself as demonstrably objective, although indeed 'relative' truths, capable of being tested once for all by the 'inconceivability of the opposite,' and certain to hold true for the whole 'knowable' universe. Thus, whether one dwelt upon the results of such a mathematical procedure as that to which M. Poincaré refers in his opening paragraphs, or whether, like Spencer himself, one applied the 'first principles' to regions of less exact science, this confidence that a certain orthodoxy regarding the principles of science was established forever was characteristic of the followers of the movement in question. Experience, lighted up by reason, seemed to them to have predetermined for all future time certain great theoretical results regarding the real constitution of the 'knowable' cosmos. Whoever doubted this doubted 'the verdict of science.'

Some of us well remember how, when Stallo's 'Principles and Theories of Modern Physics' first appeared, this sense of scientific orthodoxy was shocked amongst many of our American readers and teachers of science. I myself can recall to mind some highly authoritative reviews of that work in which the author was more or less sharply taken to task for his ignorant presumption in speaking with the freedom that he there used regarding such sacred possessions of humanity as the fundamental concepts of physics. That very book, however, has quite lately been translated into German as a valuable contribution to some of the most recent efforts to reconstitute a modern 'philosophy of nature.' And whatever may be otherwise thought of Stallo's critical methods, or of his results, there can be no doubt that, at the present moment, if his book were to appear for the first time, nobody would attempt to discredit the work merely on account of its disposition to be agnostic regarding the objective reality of the concepts of the kinetic theory of gases, or on account of its call for a logical rearrangement of the fundamental concepts of the theory of energy. We are no longer able so easily to know heretics at first sight.

For we now appear to stand in this position: The control of natural phenomena, which through the sciences men have attained, grows daily vaster and more detailed, and in its details more assured. Phenomena men know and predict better than ever. But regarding the most general theories, and the most fundamental, of science, there is no longer any notable scientific orthodoxy. Thus, as knowledge grows firmer and wider, conceptual construction becomes less rigid. The field of the theoretical philosophy of nature—yes, the field of the logic of science—this whole region is to-day an open one. Whoever will work there must indeed accept the verdict of experience regarding what happens in the natural world. So far he is indeed bound. But he may undertake without hindrance from mere tradition the task of trying afresh to reduce what happens to conceptual unity. The circle-squarers and the inventors of devices for perpetual motion are indeed still as unwelcome in scientific company as they were in the days when scientific orthodoxy was more rigidly defined; but that is not because the foundations of geometry are now viewed as completely settled, beyond controversy, nor yet because the 'persistence of force' has been finally so defined as to make the 'opposite inconceivable' and the doctrine of energy beyond the reach of novel formulations. No, the circle-squarers and the inventors of devices for perpetual motion are to-day discredited, not because of any unorthodoxy of their general philosophy of nature, but because their views regarding special facts and processes stand in conflict with certain equally special results of science which themselves admit of very various general theoretical interpretations. Certain properties of the irrational number π are known, in sufficient multitude to justify the mathematician in declining to listen to the arguments of the circle-squarer; but, despite great advances, and despite the assured results of Dedekind, of Cantor, of Weierstrass and of various others, the general theory of the logic of the numbers, rational and irrational, still presents several important features of great obscurity; and the philosophy of the concepts of geometry yet remains, in several very notable respects, unconquered territory, despite the work of Hilbert and of Pieri, and of our author himself. The ordinary inventors of the perpetual motion machines still stand in conflict with accepted generalizations; but nobody knows as yet what the final form of the theory of energy will be, nor can any one say precisely what place the phenomena of the radioactive bodies will occupy in that theory. The alchemists would not be welcome workers in modern laboratories; yet some sorts of transformation and of evolution of the elements are to-day matters which theory can find it convenient, upon occasion, to treat as more or less exactly definable possibilities; while some newly observed phenomena tend to indicate, not indeed that the ancient hopes of the alchemists were well founded, but that the ultimate constitution of matter is something more fluent, less invariant, than the theoretical orthodoxy of a recent period supposed. Again, regarding the foundations of biology, a theoretical orthodoxy grows less possible, less definable, less conceivable (even as a hope) the more knowledge advances. Once 'mechanism' and 'vitalism' were mutually contradictory theories regarding the ultimate constitution of living bodies. Now they are obviously becoming more and more 'points of view,' diverse but not necessarily conflicting. So far as you find it convenient to limit your study of vital processes to those phenomena which distinguish living matter from all other natural obects, you may assume, in the modern 'pragmatic' sense, the attitude of a 'neo-vitalist.' So far, however, as you are able to lay stress, with good results, upon the many ways in which the life processes can be assimilated to those studied in physics and in chemistry, you work as if you were a partisan of 'mechanics.' In any case, your special science prospers by reason of the empirical discoveries that you make. And your theories, whatever they are, must not run counter to any positive empirical results. But otherwise, scientific orthodoxy no longer predetermines what alone it is respectable for you to think about the nature of living substance.

This gain in the freedom of theory, coming, as it does, side by side with a constant increase of a positive knowledge of nature, lends itself to various interpretations, and raises various obvious questions.

II

One of the most natural of these interpretations, one of the most obvious of these questions, may be readily stated. Is not the lesson of all these recent discussions simply this, that general theories are simply vain, that a philosophy of nature is an idle dream, and that the results of science are coextensive with the range of actual empirical observation and of successful prediction? If this is indeed the lesson, then the decline of theoretical orthodoxy in science is—like the eclipse of dogma in religion—merely a further lesson in pure positivism, another proof that man does best when he limits himself to thinking about what can be found in human experience, and in trying to plan what can be done to make human life more controllable and more reasonable. What we are free to do as we please—is it any longer a serious business? What we are free to think as we please—is it of any further interest to one who is in search of truth? If certain general theories are mere conceptual constructions, which to-day are, and to-morrow are cast into the oven, why dignify them by the name of philosophy? Has science any place for such theories? Why be a 'neo-vitalist,' or an 'evolutionist,' or an 'atomist,' or an 'Energetiker'? Why not say, plainly: "Such and such phenomena, thus and thus described, have been observed; such and such experiences are to be expected, since the hypotheses by the terms of which we are required to expect them have been verified too often to let us regard the agreement with experience as due merely to chance; so much then with reasonable assurance we know; all else is silence—or else is some matter to be tested by another experiment?" Why not limit our philosophy of science strictly to such a counsel of resignation? Why not substitute, for the old scientific orthodoxy, simply a confession of ignorance, and a resolution to devote ourselves to the business of enlarging the bounds of actual empirical knowledge?

Such comments upon the situation just characterized are frequently made. Unfortunately, they seem not to content the very age whose revolt from the orthodoxy of traditional theory, whose uncertainty about all theoretical formulations, and whose vast wealth of empirical discoveries and of rapidly advancing special researches, would seem most to justify these very comments. Never has there been better reason than there is to-day to be content, if rational man could be content, with a pure positivism. The splendid triumphs of special research in the most various fields, the constant increase in our practical control over nature—these, our positive and growing possessions, stand in glaring contrast to the failure of the scientific orthodoxy of a former period to fix the outlines of an ultimate creed about the nature of the knowable universe. Why not 'take the cash and let the credit go'? Why pursue the elusive theoretical 'unification' any further, when what we daily get from our sciences is an increasing wealth of detailed information and of practical guidance?

As a fact, however, the known answer of our own age to these very obvious comments is a constant multiplication of new efforts towards large and unifying theories. If theoretical orthodoxy is no longer clearly definable, theoretical construction was never more rife. The history of the doctrine of evolution, even in its most recent phases, when the theoretical uncertainties regarding the 'factors of evolution' are most insisted upon, is full of illustrations of this remarkable union of scepticism in critical work with courage regarding the use of the scientific imagination. The history of those controversies regarding theoretical physics, some of whose principal phases M. Poincaré, in his book, sketches with the hand of the master, is another illustration of the consciousness of the time. Men have their freedom of thought in these regions; and they feel the need of making constant and constructive use of this freedom. And the men who most feel this need are by no means in the majority of cases professional metaphysicians—or students who, like myself, have to view all these controversies amongst the scientific theoreticians from without as learners. These large theoretical constructions are due, on the contrary, in a great many cases to special workers, who have been driven to the freedom of philosophy by the oppression of experience, and who have learned in the conflict with special problems the lesson that they now teach in the form of general ideas regarding the philosophical aspects of science.

Why, then, does science actually need general theories, despite the fact that these theories inevitably alter and pass away? What is the service of a philosophy of science, when it is certain that the philosophy of science which is best suited to the needs of one generation must be superseded by the advancing insight of the next generation? Why must that which endlessly grows, namely, man's knowledge of the phenomenal order of nature, be constantly united in men's minds with that which is certain to decay, namely, the theoretical formulation of special knowledge in more or less completely unified systems of doctrine?

I understand our author's volume to be in the main an answer to this question. To be sure, the compact and manifold teachings which this text contains relate to a great many different special issues. A student interested in the problems of the philosophy of mathematics, or in the theory of probabilities, or in the nature and office of mathematical physics, or in still other problems belonging to the wide field here discussed, may find what he wants here and there in the text, even in case the general issues which give the volume its unity mean little to him, or even if he differs from the author's views regarding the principal issues of the book. But in the main, this volume must be regarded as what its title indicates—a critique of the nature and place of hypothesis in the work of science and a study of the logical relations of theory and fact. The result of the book is a substantial justification of the scientific utility of theoretical construction—an abandonment of dogma, but a vindication of the rights of the constructive reason.

III

The most notable of the results of our author's investigation of the logic of scientific theories relates, as I understand his work, to a topic which the present state of logical investigation, just summarized, makes especially important, but which has thus far been very inadequately treated in the text-books of inductive logic. The useful hypotheses of science are of two kinds:

1. The hypotheses which are valuable *precisely* because they are either verifiable or else refutable through a definite appeal to the tests furnished by experience; and

2. The hypotheses which, despite the fact that experience suggests them, are valuable *despite,* or even *because,* of the fact that experience can *neither* confirm nor refute them. The contrast between these two kinds of hypotheses is a prominent topic of our author's discussion.

Hypotheses of the general type which I have here placed first in order are the ones which the text-books of inductive logic and those summaries of scientific method which are customary in the course of the elementary treatises upon physical science are already accustomed to recognize and to characterize. The value of such hypotheses is indeed undoubted. But hypotheses of the type which I have here named in the second place are far less frequently recognized in a perfectly explicit way as useful aids in the work of special science. One usually either fails to admit their presence in scientific work, or else remains silent as to the reasons of their usefulness. Our author's treatment of the work of science is therefore especially marked by the fact that he explicitly makes prominent both the existence and the scientific importance of hypotheses of this second type. They occupy in his discussion a place somewhat analogous to each of the two distinct positions occupied by the 'categories' and the 'forms of sensibility,' on the one hand, and by the 'regulative principles of the reason,' on the other hand, in the Kantian theory of our knowledge of nature. That is, these hypotheses which can neither be confirmed nor refuted by experience appear, in M. Poincaré's account, partly (like the conception of 'continuous quantity') as devices of the understanding whereby we give conceptual unity and an invisible connectedness to certain types of phenomenal facts which come to us in a discrete form and in a confused variety; and partly (like the larger organizing concepts of science) as principles regarding the structure of the world in its wholeness; *i. e.*, as principles in the light of which we try to interpret our experience, so as to give to it a totality and an inclusive unity such as Euclidean space, or such as the world of the theory of energy is conceived to possess. Thus viewed, M. Poincaré's logical theory of this second class of hypotheses undertakes to accomplish, with modern means and in the light of to-day's issues, a part of what Kant endeavored to accomplish in his theory of scientific knowledge with the limited means which were at his disposal. Those aspects of science which are determined by the use of the hypotheses of this second kind appear in our author's account as constituting an essential human way of viewing nature, an interpretation rather than a portrayal or a prediction of the objective facts of nature, an adjustment of our conceptions of things to the internal needs of our intelligence, rather than a grasping of things as they are in themselves.

To be sure, M. Poincaré's view, in this portion of his work, obviously differs, meanwhile, from that of Kant, as well as this agrees, in a measure, with the spirit of the Kantian epistemology. I do not mean therefore to class our author as a Kantian. For Kant, the interpretations imposed by the 'forms of sensibility,' and by the 'categories of the understanding,' upon our doctrine of nature are rigidly predetermined by the unalterable 'form' of our intellectual powers. We 'must' thus view facts, whatever the data of sense must be. This, of course, is not M. Poincaré's view. A similarly rigid predetermination also limits the Kantian 'ideas of the reason' to a certain set of principles whose guidance of the course of our theoretical investigations is indeed only 'regulative,' but is 'a priori,' and so unchangeable. For M. Poincaré, on the contrary, all this adjustment of our interpretations of experience to the needs of our intellect is something far less rigid and unalterable, and is constantly subject to the suggestions of experience. We must indeed interpret in our own way; but our way is itself only relatively determinate; it is essentially more or less plastic; other interpretations of experience are conceivable. Those that we use are merely the ones found to be most convenient. But this convenience is not absolute necessity. Unverifiable and irrefutable hypotheses in science are indeed, in general, indispensable aids to the organization and to the guidance of our interpretation of experience. But it is experience itself which points out to us what lines of interpretation will prove most convenient. Instead of Kant's rigid list of *a priori* 'forms,' we consequently have in M. Poincaré's account a set of conventions, neither wholly subjective and arbitrary, nor yet imposed upon us unambiguously by the external compulsion of experience. The organization of science, so far as this organization is due to hypotheses of the kind here in question, thus resembles that of a constitutional government—neither absolutely necessary, nor yet determined apart from the will of the subjects, nor yet accidental—a free, yet not a capricious establishment of good order, in conformity with empirical needs.

Characteristic remains, however, for our author, as, in his decidedly contrasting way, for Kant, the thought that *without principles which at every stage transcend precise confirmation through such experience as is then accessible the organization of experience is impossible.* Whether one views these principles as conventions or as *a priori* 'forms,' they may therefore be described as hypotheses, but as hypotheses that, while lying at the basis of our actual physical sciences, at once refer to experience and help us in dealing with experience, and are yet neither confirmed nor refuted by the experiences which we possess or which we can hope to attain.

Three special instances or classes of instances, according to our author's account, may be used as illustrations of this general type of hypotheses. They are: (1) The hypothesis of the existence of continuous extensive *quanta* in nature; (2) The principles of geometry; (3) The principles of mechanics and of the general theory of energy. In case of each of these special types of hypotheses we are at first disposed, apart from reflection, to say that we *find* the world to be thus or thus, so that, for instance, we can confirm the thesis according to which nature contains continuous magnitudes; or can prove or disprove the physical truth of the postulates of Euclidean geometry; or can confirm by definite experience the objective validity of the principles of mechanics. A closer examination reveals, according to our author, the incorrectness of all such opinions. Hypotheses of these various special types are needed; and their usefulness can be empirically shown. They are in touch with experience; and that they are not merely arbitrary conventions is also verifiable. They are not *a priori* necessities; and we can easily conceive intelligent beings whose experience could be best interpreted without using these hypotheses. Yet these hypotheses are *not* subject to direct confirmation or refutation by experience. They stand then in sharp contrast to the scientific hypotheses of the other, and more frequently recognized, type, *i. e.,* to the hypotheses which *can* be tested by a definite appeal to experience. To these other hypotheses our author attaches, of course, great importance. His treatment of them is full of a living appreciation of the significance of empirical investigation. But the central problem of the logic of science thus becomes the problem of the relation between the two fundamentally distinct types of hypotheses, *i. e.*, between those which can not be verified or refuted through experience, and those which can be empirically tested.

IV

The detailed treatment which M. Poincaré gives to the problem thus defined must be learned from his text. It is no part of my purpose to expound, to defend or to traverse any of his special conclusions regarding this matter. Yet I can not avoid observing that, while M. Poincaré strictly confines his illustrations and his expressions of opinion to those regions of science wherein, as special investigator, he is himself most at home, the issues which he thus raises regarding the logic of science are of even more critical importance and of more impressive interest when one applies M. Poincaré's methods to the study of the concepts and presuppositions of the organic and of the historical and social sciences, than when one confines one's attention, as our author here does, to the physical sciences. It belongs to the province of an introduction like the present to point out, however briefly and inadequately, that the significance of our author's ideas extends far beyond the scope to which he chooses to confine their discussion.

The historical sciences, and in fact all those sciences such as geology, and such as the evolutionary sciences in general, undertake theoretical constructions which relate to past time. Hypotheses relating to the more or less remote past stand, however, in a position which is very interesting from the point of view of the logic of science. Directly speaking, no such hypothesis is capable of confirmation or of refutation, because we can not return into the past to verify by our own experience what then happened. Yet indirectly, such hypotheses may lead to predictions of coming experience. These latter will be subject to control. Thus, Schliemann's confidence that the legend of Troy had a definite historical foundation led to predictions regarding what certain excavations would reveal. In a sense somewhat different from that which filled Schliemann's enthusiastic mind, these predictions proved verifiable. The result has been a considerable change in the attitude of historians toward the legend of Troy. Geological investigation leads to predictions regarding the order of the strata or the course of mineral veins in a district, regarding the fossils which may be discovered in given formations, and so on. These hypotheses are subject to the control of experience. The various theories of evolutionary doctrine include many hypotheses capable of confirmation and of refutation by empirical tests. Yet, despite all such empirical control, it still remains true that whenever a science is mainly concerned with the remote past, whether this science be archeology, or geology, or anthropology, or Old Testament history, the principal theoretical constructions always include features which no appeal to present or to accessible future experience can ever definitely test. Hence the suspicion with which students of experimental science often regard the theoretical constructions of their confrères of the sciences that deal with the past. The origin of the races of men, of man himself, of life, of species, of the planet; the hypotheses of anthropologists, of archeologists, of students of 'higher criticism'—all these are matters which the men of the laboratory often regard with a general incredulity as belonging not at all to the domain of true science. Yet no one can doubt the importance and the inevitableness of endeavoring to apply scientific method to these regions also. Science needs theories regarding the past history of the world. And no one who looks closer into the methods of these sciences of past time can doubt that verifiable and unverifiable hypotheses are in all these regions inevitably interwoven; so that, while experience is always the guide, the attitude of the investigator towards experience is determined by interests which have to be partially due to what I should call that 'internal meaning,' that human interest in rational theoretical construction which inspires the scientific inquiry; and the theoretical constructions which prevail in such sciences are neither unbiased reports of the actual constitution of an external reality, nor yet arbitrary constructions of fancy. These constructions in fact resemble in a measure those which M. Poincaré in this book has analyzed in the case of geometry. They are constructions molded, but *not* predetermined in their details, by experience. We report facts; we let the facts speak; but we, as we investigate, in the popular phrase, 'talk back' to the facts. We interpret as well as report. Man is not merely made for science, but science is made for man. It expresses his deepest intellectual needs, as well as his careful observations. It is an effort to bring internal meanings into harmony with external verifications. It attempts therefore to control, as well as to submit, to conceive with rational unity, as well as to accept data. Its arts are those directed towards self-possession as well as towards an imitation of the outer reality which we find. It seeks therefore a disciplined freedom of thought. The discipline is as essential as the freedom; but the latter has also its place. The theories of science are human, as well as objective, internally rational, as well as (when that is possible) subject to external tests.

In a field very different from that of the historical sciences, namely, in a science of observation and of experiment, which is at the same time an organic science, I have been led in the course of some study of the history of certain researches to notice the existence of a theoretical conception which has proved extremely fruitful in guiding research, but which apparently resembles in a measure the type of hypotheses of which M. Poincaré speaks when he characterizes the principles of mechanics and of the theory of energy. I venture to call attention here to this conception, which seems to me to illustrate M. Poincaré's view of the functions of hypothesis in scientific work.

The modern science of pathology is usually regarded as dating from the earlier researches of Virchow, whose 'Cellular Pathology' was the outcome of a very careful and elaborate induction. Virchow, himself, felt a strong aversion to mere speculation. He endeavored to keep close to observation, and to relieve medical science from the control of fantastic theories, such as those of the *Naturphilosophen* had been. Yet Virchow's researches were, as early as 1847, or still earlier, already under the guidance of a theoretical presupposition which he himself states as follows: "We have learned to recognize," he says, "that diseases are not autonomous organisms, that they are no entities that have entered into the body, that they are no parasites which take root in the body, but *that they merely show us the course of* *the vital processes under altered conditions*" ('dass sie nur Ablauf der Lebenserscheinungen unter veränderten Bedingungen darstellen').

The enormous importance of this theoretical presupposition for all the early successes of modern pathological investigation is generally recognized by the experts. I do not doubt this opinion. It appears to be a commonplace of the history of this science. But in Virchow's later years this very presupposition seemed to some of his contemporaries to be called in question by the successes of recent bacteriology. The question arose whether the theoretical foundations of Virchow's pathology had not been set aside. And in fact the theory of the parasitical origin of a vast number of diseased conditions has indeed come upon an empirical basis to be generally recognized. Yet to the end of his own career Virchow stoutly maintained that in all its essential significance his own fundamental principle remained quite untouched by the newer discoveries. And, as a fact, this view could indeed be maintained. For if diseases proved to be the consequences of the presence of parasites, the diseases themselves, so far as they belonged to the diseased organism, were still not the parasites, but were, as before, the reaction of the organism to the *veränderte Bedingungen* which the presence of the parasites entailed. So Virchow could well insist. And if the famous principle in question is only stated with sufficient generality, it amounts simply to saying that if a disease involves a change in an organism, and if this change is subject to law at all, then the nature of the organism and the reaction of the organism to whatever it is which causes the disease must be understood in case the disease is to be understood.

For this very reason, however, Virchow's theoretical principle in its most general form *could be neither confirmed nor refuted by experience*. It would remain empirically irrefutable, so far as I can see, even if we should learn that the devil was the true cause of all diseases. For the devil himself would then simply predetermine the *veränderte Bedingungen* to which the diseased organism would be reacting. Let bullets or bacteria, poisons or compressed air, or the devil be the *Bedingungen* to which a diseased organism reacts, the postulate that Virchow states in the passage just quoted will remain irrefutable, if only this postulate be interpreted to meet the case. For the principle in question merely says that whatever entity it may be, bullet, or poison, or devil, that affects the organism, the disease is not that entity, but is the resulting alteration in the process of the organism.

I insist, then, that this principle of Virchow's is no trial supposition, no scientific hypothesis in the narrower sense—capable of being submitted to precise empirical tests. It is, on the contrary, a very precious *leading idea,* a theoretical interpretation of phenomena, in the light of which observations are to be made—'a regulative principle' of research. It is equivalent to a resolution to search for those detailed connections which link the processes of disease to the normal process of the organism. Such a search undertakes to find the true unity, whatever that may prove to be, wherein the pathological and the normal processes are linked. Now without some such leading idea, the cellular pathology itself could never have been reached; because the empirical facts in question would never have been observed. Hence this principle of Virchow's was indispensable to the growth of his science. Yet it was not a verifiable and not a refutable hypothesis. One value of unverifiable and irrefutable hypotheses of this type lies, then, in the sort of empirical inquiries which they initiate, inspire, organize and guide. In these inquiries hypotheses in the narrower sense, that is, trial propositions which are to be submitted to definite empirical control, are indeed everywhere present. And the use of the other sort of principles lies wholly in their application to experience. Yet without what I have just proposed to call the 'leading ideas' of a science, that is, its principles of an unverifiable and irrefutable character, suggested, but not to be finally tested, by experience, the hypotheses in the narrower sense would lack that guidance which, as M. Poincaré has shown, the larger ideas of science give to empirical investigation.

V

I have dwelt, no doubt, at too great length upon one aspect only of our author's varied and well-balanced discussion of the problems and concepts of scientific theory. Of the hypotheses in the narrower sense and of the value of direct empirical control, he has also spoken with the authority and the originality which belong to his position. And in dealing with the foundations of mathematics he has raised one or two questions of great philosophical import into which I have no time, even if I had the right, to enter here. In particular, in speaking of the essence of mathematical reasoning, and of the difficult problem of what makes possible novel results in the field of pure mathematics, M. Poincaré defends a thesis regarding the office of 'demonstration by recurrence'—a thesis which is indeed disputable, which has been disputed and which I myself should be disposed, so far as I at present understand the matter, to modify in some respects, even in accepting the spirit of our author's assertion. Yet there can be no doubt of the importance of this thesis, and of the fact that it defines a characteristic that is indeed fundamental in a wide range of mathematical research. The philosophical problems that lie at the basis of recurrent proofs and processes are, as I have elsewhere argued, of the most fundamental importance.

These, then, are a few hints relating to the significance of our author's discussion, and a few reasons for hoping that our own students will profit by the reading of the book as those of other nations have already done.

Of the person and of the life-work of our author a few words are here, in conclusion, still in place, addressed, not to the students of his own science, to whom his position is well known, but to the general reader who may seek guidance in these pages.

Jules Henri Poincaré was born at Nancy, in 1854, the son of a professor in the Faculty of Medicine at Nancy. He studied at the École Polytechnique and at the École des Mines, and later received his doctorate in mathematics in 1879. In 1883 he began courses of instruction in mathematics at the École Polytechnique; in 1886 received a professorship of mathematical physics in the Faculty of Sciences at Paris; then became member of the Academy of Sciences at Paris, in 1887, and devoted his life to instruction and investigation in the regions of pure mathematics, of mathematical physics and of celestial mechanics. His list of published treatises relating to various branches of his chosen sciences is long; and his original memoirs have included several momentous investigations, which have gone far to transform more than one branch of research. His presence at the International Congress of Arts and Science in St. Louis was one of the most noticeable features of that remarkable gathering of distinguished foreign guests. In Poincaré the reader meets, then, not one who is primarily a speculative student of general problems for their own sake, but an original investigator of the highest rank in several distinct, although interrelated, branches of modern research. The theory of functions—a highly recondite region of pure mathematics—owes to him advances of the first importance, for instance, the definition of a new type of functions. The 'problem of the three bodies,' a famous and fundamental problem of celestial mechanics, has received from his studies a treatment whose significance has been recognized by the highest authorities. His international reputation has been confirmed by the conferring of more than one important prize for his researches. His membership in the most eminent learned societies of various nations is widely extended; his volumes bearing upon various branches of mathematics and of mathematical physics are used by special students in all parts of the learned world; in brief, he is, as geometer, as analyst and as a theoretical physicist, a leader of his age.

Meanwhile, as contributor to the philosophical discussion of the bases and methods of science, M. Poincaré has long been active. When, in 1893, the admirable *Revue de Métaphysique et de Morale* began to appear, M. Poincaré was soon found amongst the most satisfactory of the contributors to the work of that journal, whose office it has especially been to bring philosophy and the various special sciences (both natural and moral) into a closer mutual understanding. The discussions brought together in the present volume are in large part the outcome of M. Poincaré's contributions to the *Revue de Métaphysique et de Morale*. The reader of M. Poincaré's book is in presence, then, of a great special investigator who is also a philosopher.

Chapter XIV

BENNO ERDMANN'S LOGIC

Editor's Note: This is a review of Benno Erdmann's Logische Elementarlehre. Halle. 1892. It is reprinted by permission from Philosophical Review Volume I, pp. 547-552.

This first volume of Professor Erdmann's Logic is extremely rich in content, and, like several other recent logical treatises, such as Sigwart's and Wundt's, it defies a summary judgment, and must in the main be tested by long usage. The noted student of Kant's philosophical development does here no injustice to his well-earned reputation for minuteness and carefulness of scholarship. Where close examination is required, Erdmann does not spare pains, nor does he fear to weary his readers. On the other hand, breadth of view is secured by two devices, not unfamiliar in themselves, but here carried out with great industry. The one device is that of the historical comparison of logical doctrines, the other that of great variety in the choice of concrete examples of logical forms, principles, and processes. The historical comparisons are, to be sure, very briefly set forth, in summary paragraphs, usually placed at the conclusion of each new positive statement of Erdmann's own views. The examples, on the other hand, are sometimes almost capriciously multiplied. Yet everywhere a marvellous range of literary and historical knowledge is shown, and logical authors long since almost forgotten are brought down from dusty shelves to illuminate, often with surprising vividness, our author's argument. The recent progress of logical discussion, since Ueberweg and Lotze, is also borne in mind; nearly all the "burning questions" of the logic of the past two decades are touched upon; our author has his views concerning the nature of negative propositions, concerning the "impersonals," concerning "existential" judgments, concerning hypothetical judgments, in short concerning the favorite problems of modern continental logic in general.

Characteristic for our author's whole attitude is, meanwhile, his position with regard to the "Algebra of Logic," which Schroeder has recently so well introduced to German readers. Erdmann postpones until a later volume (so I understand his statement on p. 431) "die grundsätzliche Auseinandersetzung mit diesem Formalismus, der dem wissenschaftlichen Gebrauch des Denkens fremd ist, und fremd bleiben muss," but he everywhere condemns its method. Boole's algebraic formulation of the principle of contradiction, in its relation to the formula

$x = x^2, \therefore x(1-x) = 0$, involves (p. 367) one of the "verwunderlichsten Irrungen der mathematischen Logik," and "needs no criticism." There are a number of references of the same general character scattered through this volume. Nor is the reason of Erdmann's antipathy far to seek. He is, namely, a pronounced opponent of the "*Subsumtionstheorie*" of the judgment. And the theory of the judgment is, as he tells us already in his preface, the "Brennpunkt der Logik." The geometrical method of symbolizing judgments is the natural expression (p. 247, p. 446 *sq.*) of this false subsumption theory, and, "roh" as it is, "may be of some use to the beginner." But Erdmann loses no opportunity to warn us against any extension or elaboration of it. In his dislike of the "mathematisirende Logik," and of its "formalism," Erdmann reminds us in fact of Hegel's attitude towards so much of the mathematical tendency as existed in the logic of his own time, and one is rather surprised to find Erdmann himself putting Hegel (pp. 247, 248) in the wrong company, in consequence of the latter's account of "das abstrakte Urtheil." In many respects, as a fact, Erdmann's notion of the nature of the thinking process brings him fairly near to Hegel, although in other respects the two are very far apart.

Our author's personal theory of the judgment allies him, meanwhile, to the teachers of the "Logic of Intension" in general. Intension determines extension (p. 151). The extension is the "totality of the species in a genus" (p. 134), but has nothing to do with the "Anzahl der einzelnen Exemplare" of the genus in question. The extension of the class *match*, for example, is not decreased when you burn this individual match, or when you manufacture new matches of an old sort (*loc. cit.*). Not only is intension thus prior to extension, and of far deeper significance than the latter, but, in view of the rejection of all forms of the "subsumption theory," we are driven (p. 261) to define the judgment as a "Gleichheitsbeziehung der Einordnung" — a technical expression which needs some elucidation, but which is at all events the embodiment of an interpretation of the process of judgment in terms of intension. The predicate of a judgment points out elements or groups of elements which, intensively regarded, *were already immanent* in the idea of the subject. Such is Erdmann's view.

Our elucidation of the foregoing expression must needs be inadequate, since our author's intricate argument defies successful condensation. But it is upon this "focal" point that all the rays of this treatise are indeed brought to bear; and a hint of the nature of this doctrine involves of necessity a characterization of the whole book. And such a characterization, inadequate as it must needs be, we have here to undertake.

In general, in his opening paragraphs, Erdmann defines Logic as the "science of the formal presuppositions of valid judgments concerning the objects of inner and outer perception" (p. 15). As *formal*, the presuppositions which logic studies, *i.e.* the presuppositions of "valid," or of "scientific" thinking, are to be distinguished from those *material* presuppositions which are the topic, on the one hand of metaphysics, on the other hand of special science. Logic is then (p. 16) a general or formal "normative" science of the thinking process. Its office is thus indeed different from that of the psychology of the thinking process (p. 18); but it is impossible to study norms without understanding actual processes, and if logic (p. 19) is to avoid barren schematism, the logician must base his study upon a psychological analysis of the natural history of the thinking process. To such a natural history Erdmann devotes, in fact, considerable space. The first book of his treatise (pp. 35–186), on the "*Gegenstände des Denkens*," contains much psychological material. By "objects of thought," Erdmann means the sum total of the "*Vorgestelltes*," *i.e.* of that of which we have ideas (p. 81). In his own metaphysic disposed to realism, Erdmann endeavors as far as possible to separate logical and metaphysical problems (pp. 10–12, 77, 81–85). The "*Gegenstände der Vorstellungen*" or "*des Denkens*" are therefore not, in general, for the logician, the things in themselves, or "*das Transcendente*" (p. 10), but the "objects whose elements are given to us in inner or outer perception" (p. 12). These objects it is that in science we are directly thinking about. "*Das Transcendente*" we conceive only indirectly through the objects of inner or outer experience, and it is in regard to the latter that there arises the general question of Logic (p. 12), *viz.*: "What is our right to assume the possibility of valid judgment concerning '*das Vorgestellte*'?" "*Das Vorgestellte*" itself consists either of "original objects" or of "derived objects" (p. 38, "*ursprüngliche und abgeleitete Gegenstände*"). The former correspond to Locke's "ideas" of outer and inner sense. The latter include the ideas of memory, and *die abstrakten Vorstellungen*. As for Erdmann's use of *Vorstellung*, as he explains it on p. 36, the word relates to "all contents of consciousness in which objects are presented," and "unconscious *Vorstellungen*" are self-contradictions. On the other hand, however, the process of *Apperception* (the word being used in the Herbartian sense) involves mental processes which themselves remain in large part unconscious (p. 42), and Erdmann makes considerable use of the category of the unconscious in mental life, his view of this matter coloring important logical analyses in the course of the book (*cf.* p. 77, and the argument on p. 210 with regard to Kant's doctrine of the "synthetic" judgments of perception). On the basis of this general theory of Apperception, Erdmann introduces, on p. 45, a very characteristic doctrine concerning the process of abstraction. In abstraction, namely, one does not really get rid of the diverse features of individual objects and retain only the like elements as giving an idea of a class. The association of the like elements, in a series of objects of experience, with the unlike elements, remains for our consciousness actually the same as at first, except that it is "not so close" as it would be in case one conceived a single individual in isolation. Consciousness is, for our author, always concrete and synthetic. Our relative abstractions we accomplish, in the more complex cases, only by virtue of bringing into consciousness a series of *Vorstellungen* whose common features are strengthened through *Verschmelzung*, while the individual features of each *Vorstellung* are kept in the background by virtue of that very flow of consciousness which helps us to attend to the common features themselves. Abstraction is an affair of attention (p. 48). It does not really sunder, it emphasizes; and for its emphasis it is largely dependent upon the flow of consciousness, which presents the *Vorstellungen* that are to be the object of the abstracting process.

A passage from this theory of abstraction to the theory of judgment, as indicated above, is contained in our author's doctrine of the significance of language for the thinking process. Abstract ideas are made possible by means of "sprachlicher Ueberlieferung" (p. 49). The imagination is excited to the formation of abstract ideas by the aid and the usage of language (p. 51). In particular the whole activity of judgment depends upon language (p. 20 *sqq.*). Perception and mere *Vorstellen* can go on without speech; but judgment, which (p. 1) is the essential characteristic of the thinking process ("unter Denken soll nichts anderes als Urtheilen verstanden werden"), is impossible without some sort of language (pp. 23, 234). Erdmann undertakes more than once, and at considerable length, a psychological proof of this assertion; but admits (p. 224) that a final *experimentum crucis* is still lacking to demonstrate his position, and no doubt wisely rejects the evidence offered by those cases of *aphasia* which have been observed and cited with respect to this problem, as insufficient to prove any unmistakable result. Our author is meanwhile far from supposing that the language process is *identical* with the thinking process, and the observations on pp. 229–231 concerning the relations of the two processes are very instructive, even if one questions Erdmann's hypotheses concerning the "unconscious" side of the mental life at moments of reading or of hearing a discourse.

Judgment then depends upon language. The data of inner and outer perception, organized through the apperceptive process, give us an enormously complex *Vorstellungsleben*, where the *Verschmelzung* of similar elements in series of allied ideas gives rise to multitudinous sorts and groupings of *Abstrakte Vorstellungen.* These in their turn are formed, held, and communicated under the influence of language. Thinking, however, is judging concerning these results of our experience and of our apperceptive process. In what does this act of judgment consist? Erdmann replies: Not in bringing together already formed ideas and uniting them two and two by a fresh "function of unity"; not in subsuming one under another; not in uniting mere names of things; not even in identifying ideas, nor yet in sundering ideas already united. On the contrary, the *Wahrnehmungsurtheil*, the simplest case of all, already typifies (p. 205) the essential nature of the process of judgment. When I say, *This paper is four-cornered*, I do not sunder the object of my sight into two thought-objects; nor do I bring into a new union two significant ideas before sundered. Neither mere analysis nor mere synthesis takes place here. But (p. 203) while the paper remains all the while just as it is for my consciousness, I bring in succession ideas of words, *Wortvorstellungen*, into my mind, and observe that two of these words, *i.e.* the pair in the compound *four-cornered*, express a meaning which *in this or that respect is identical with some of the facts already presented in my one and indivisible perception of the paper.* My perception then is, if you will, a synthesis in experience, a *Verflechtung*, of many perceptual facts. Into the unity of this perception my thought introduces, according to Erdmann's theory, no sundering whatever. What was united in the *Vorstellung* before I judged, remains united while I judge, and stays united afterwards. The new thing that happens while I judge is for the first a *Vorstellungsverlauf*, consisting of words. I observe meanwhile that these words express meanings which are identical with something that is already immanent in the perceptive unity itself. My judgment is thus a comment in successive *Wortvorstellungen* upon what already coexists in unity in the subject of my judgment. As such a comment my judgment finds predicates for this subject, but does not change the content of the latter.

The "predicative relation" is in general thus typified. The predicate is in meaning discovered to be immanent in the subject. This discovery is what the judgment accomplishes. The predicate is represented by a *Wortvorstellung*, which in so far comes to the subject from without. Yet the act of judgment does not create, but only finds the unity of the subject and predicate idea, and finds this unity as having been already existent in the subject before the judgment was made. Here is then some indication of what Erdmann means by his *Gleichheitsbeziehung der Einordnung*, which one may freely translate as the "discovery of an identity between the meaning of the predicate and a portion of the meaning already immanent in the subject."

The relation of this view of the thinking process to Erdmann's former insistence upon the inseparability of the abstractly common and the individually peculiar elements of the objects of our generalizing consciousness, is not hard to see. There is considerable difficulty, of course, in carrying through such a notion of the process of judgment as this one in case of the more abstract and complex forms of judgment, *e.g.* in case of judgments of relation, and in the case of seemingly purely constructive judgments, such as definitions. But our author works patiently, and with much success. Negative judgments he regards with Sigwart as rejections of attempted positive judgments, and not as themselves a species of simple judgment (pp. 349–363). The impersonals proper are simply judgments whose subjects are left very indefinite (p. 307). On the whole (p. 262) he defines a judgment as "The inclusion of one object (of consciousness) in the (intensive) content of another, — this inclusion (1) being conceived as in logical immanence, (2) being determined by the identity of content of the material constituents, and (3) being expressed in a proposition."

There is here no space to follow our author into the applications of this theory. One is not surprised to find that, with his eyes fixed upon so interesting a psychological problem, he should almost wholly neglect the considerations that to many of us make the 'Algebra of Logic' so promising and important a region of exact inquiry; nor are most philosophical students likely to be satisfied with the sceptical discussion of the nature of logical necessity (pp. 372–378); and the observations upon probability, mathematical and non-mathematical (pp. 388 *sqq.*), have appeared to the present writer especially unsatisfactory. But if these matters seem to us to indicate our author's limitations, we have to thank him on the whole for a most learned and stimulating study of the problems of philosophical logic, and particularly of the problem of the judgment.

The discussion of the syllogism is extended, and full of interesting matter. A later volume is to be devoted to a general Doctrine of Method.

Part II

Symbolic Logic

Chapter XV

AN EXTENSION OF THE ALGEBRA OF LOGIC

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BY the "algebra of logic" I mean, for the purposes of the present paper, Boole's calculus, in its classic form. I shall ignore, for the time, all the applications of this calculus to "classes," to "propositions," or to any other special sorts of objects. By the "Boolean entities" I shall here mean simply whatever entities conform to the laws which Boole's algebra expresses. There are such entities,—for instance, "classes," and "propositions," as well as "areas." But there are other Boolean entities than these; and I am here concerned only with Boolean entities abstractly viewed.

I

Considered as an algebra, the Boolean calculus, as is well known, occupies an unique but disappointing place among the algebras known to modern mathematics. In opening two remarkable contributions to the algebra of logic which were published a few years ago[1] Whitehead said that Boole's algebra might be compared with the chemical element argon. For, as Whitehead remarked, Boole's calculus had so far refused to form compounds with other equally elementary theories. Whitehead himself undertook, in the papers in question, to contribute towards the needed change of this unprofitable state of isolation. But, important as his results were, Whitehead has not, so far as I know, published further researches upon the same topic. The problem remains still on our hands: Can any one discover devices to make this argon among the pure algebras somewhat better disposed to unite with its fellow-algebras?

This question may seem to have a merely formal interest: and in this paper I shall deal wholly with formalities. Yet, as I hope to show in future papers, very interesting philosophical issues are bound up with the answer to the question which Whitehead's comparison of the Boolean calculus to argon presents to our notice. This is a region of philosophy where some of the most abstract, and some of the deepest and richest of philosophical interests lie very near together.

[1] See *American Journal of Mathematics,* Vol. 23, pages 139, 297.

II

The Nature of the Boolean Operations

The general reason why Boole's calculus has proved so austerely unproductive of the mathematical novelties for which Boole himself hoped, is well known. The fundamental operations of the Boolean calculus, *viz.*, the "addition" and the "multiplication" which characterize this algebra, appear, at first sight, to promise notable new combinations, since, like the corresponding operations in ordinary algebra, they are both commutative and associative. Furthermore, each of these Boolean operations is distributive with reference to the other. Their dual relation to "negation," as expressed in "De Morgan's theorem," is a very attractive property, which especially helps to give to this algebra its unique place amongst forms of symbolism. But alas, neither the "multiplication" nor the "addition" of this algebra is unconditionally invertible. Nor is the result of such inversion, when there is such a result, free from ambiguity. "Division" and "subtraction" have existence only subject to exasperating restrictions; and have never received any really notable development. In brief, the fundamental operations of the algebra are not group-operations. Hence the theory of groups is, except for one striking, but, so far, comparatively unfruitful exception, inapplicable to the Boolean algebra in its classic form. But without group-operations how shall an algebra progress?

III

The Operation of Jevons, the Prime-Functions of Whitehead, and the T-Relation

The exception just mentioned, the one group-operation which the Boolean calculus in its original form permits, was first noticed by Jevons. Schroeder and, still later, Whitehead, have dealt with it at considerable length. I have here only a word to add to the observations which Schroeder and Whitehead have made upon this particular topic; but this word relates, I believe, to a logical relation whose philosophical importance may appear, as I hope, in papers which I hope later to prepare.

Jevons noted that the three Boolean equations:—

$$a\bar{b}+\bar{a}b=c,$$
$$b\bar{c}+\bar{b}c=a,$$
$$a\bar{c}+\bar{a}c=b,$$

express precisely equivalent propositions, since each of them follows from either of the others. Schroeder and Whitehead have in somewhat different ways expressed the very natural observation that this equivalence of the three equations of Jevons defines a group-operation.

Let $a\bar{b}+\bar{a}b=(a \circ b)$, where the symbol $\circ$ stands for an operation performed upon the elements of the pair (a, b). Then, if we suppose that this operation is viewed as already known,—we observe that whatever pair of the "logical entities" of the Boolean calculus we may choose, the entity $(a \circ b)$ is itself a perfectly determinate Boolean logical entity. The operation $\circ$ is commutative and associative. The three equations of Jevons show that this operation is not only invertible, but is also its own inverse, so that the three symbolic expressions, $(a \circ b)=c$, $(a \circ c)=b$, $(b \circ c)=a$, express mutually equivalent propositions. The group defined by this operation is the "axial group," as has been pointed out by Professor Miller, of the University of Illinois, to whom I owe this last observation.

Whitehead has proposed to call functions whose form is that of $a\bar{b}+\bar{a}b$, "prime functions" or simply "primes." For a reason, explained in his papers,[2] Whitehead also calls these functions "primary primes."

The equation: $a\bar{b}+\bar{a}b=c$ appears, at first sight, to be expressive of a triadic relation of the elements (a, b, c). But as early as 1905 I myself noted that the relation involved is in fact *tetradic*. This latter observation was at that time, I think, new; and I have never heretofore printed the very obvious statements regarding this tetradic relation which here follow, and which I have constantly used in lectures ever since 1905. Some fellow-students may still find novel the form of expression which I employ.

Let us suppose true the equation:—

$$a\bar{b}+\bar{a}b=c\bar{d}+\bar{c}d.$$

Then the four elements (a, b, c, d) stand in a symmetrical tetradic relation which can be expressed by solving the equation, successively, for a, for b, for c, and for d. Thus,

$$a=(bc+\bar{b}\bar{c})d+(b\bar{c}+\bar{b}c)\bar{d};$$
$$b=(ad+\bar{a}\bar{d})c+(a\bar{d}+\bar{a}d)\bar{c};$$
$$=(ac+\bar{a}\bar{c})d+(a\bar{c}+\bar{a}c)\bar{d}.$$

That is, in case the equation: $a\bar{b}+\bar{a}b=c\bar{d}+\bar{c}d$, is true, each of the four elements of the tetrad (a, b, c, d) is a determinate symmetrical function of the three remaining elements, while the form of this function remains the same, whatever one of the four elements we choose to express in terms of the others. I propose to symbolize the totally symmetrical tetradic relation here in question by the relational form $T(abcd)$, which is to be read as a proposition: "The four

[2] See *American Journal of Mathematics*, Vol. 23, page 147.

elements (a, b, c, d) stand in the four-term relation T." The proposition $T(abcd)$ is thus precisely equivalent to any one of the equations:

$$a\bar{b}+\bar{a}b=c\bar{d}+\bar{c}d;\ a\bar{c}+\bar{a}c=b\bar{d}+\bar{b}d, \text{ etc.}$$

The properties of this T-relation are then the following:—

1. The relation is (as just pointed out) totally symmetrical. That is, $T(abcd)=T(cdba)=T(dcab)$, etc.

2. Given (a, b, c) or, in fact, given any triad of the elements of the tetrad (a, b, c, d), chosen quite without restriction,—then, by the requirement that $T(abcd)$ shall be true, the fourth element of the tetrad is uniquely determined.

3. The T-relation $T(abcd)$ remains invariant in case, for any *two* of the elements of the tetrad, we substitute their respective negatives. Thus $T(abcd)=T(ab\bar{c}\bar{d})=T(\bar{a}b\bar{c}d)$, etc.

4. The T-relation is *transitive by pairs*. That is, if $T(abcd)$, and if $T(abef)$, it follows that $T(cdef)$. For $a\bar{b}+\bar{a}b=c\bar{d}+\bar{c}d=e\bar{f}+\bar{e}f$.

5. If $T(abcd)$ and if $a=b$, then it follows that $c=d$; if $b=c$, then $a=d$, etc. Conversely, $T(aabb)$, $T(\bar{a}\bar{a}bb)$, $T(aaaa)$, are true propositions, whatever Boolean elements a and b may be.

If, hereupon, we make the logical element 0 a member of a tetrad, and if $T(abc0)$ is true, this fact may also be expressed by any one of the three equations of Jevons:—

$$a\bar{b}+\bar{a}b=c,\ b\bar{c}+\bar{b}c=a,\ a\bar{c}+\bar{a}c=b.$$

It follows that equations involving the "prime-functions" of Whitehead can be expressed in terms of T-relations; and that, in particular, the equation: $a\bar{b}+\bar{a}b=c$ should be regarded, *not* as simply expressive of a triadic relation of (a, b, c), but rather as expressing a symmetrical tetradic relation of a, b, c, and 0.

This, the only group-operation of the classical Boolean algebra, may thus be regarded as deriving its properties from those of the T-relation. The total symmetry of this tetradic relation is responsible for the simplicity of the group in question, and for its comparative unfruitfulness as a source of novelties in the Boolean calculus.

I shall have occasion to return to the T-relation in various stages of the inquiry which is herewith begun.

IV

The Ordinal Functions

So far we have traversed what is, on the whole, decidedly familiar ground. From this point onwards we shall deal with matters which I believe to be novel.

There exists a class of functions, in the algebra of logic, to which, since 1909, when I first observed some of their more interesting properties, I have devoted a good deal of study. Here is not yet the place to tell why these functions are as important for logical theory as I believe them to be. My present task will be limited to defining these functions, and to showing that, by making a due use of them, we can easily lay the foundation for a very considerable extension of the classical Boolean calculus. I shall give to the functions in question the name "ordinal functions."

Every student of Boole's algebra is familiar with functions of the form:—

$$p = axy + bx\bar{y} + c\bar{x}y + d\bar{x}\bar{y}.$$

It occasionally happens, in dealing with the solution of equations, that one may meet with the special case where $a = \bar{d}$, $b = \bar{c}$. The foregoing function then becomes:—

$$\begin{aligned} p &= axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} \\ &= abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y}. \end{aligned}$$

If $p = 0$, the resulting equation has an interesting pair of roots. For by virtue of the foregoing transformation, which the special function here in question permits, the two unknowns (x, y) in the equation:—

$$axy + bx\bar{y} + \bar{b}\bar{x}y + \bar{a}\bar{x}\bar{y} = 0,$$

can be separated for the purpose of solution. For we have, as above:

$$abx + \bar{a}\bar{b}\bar{x} + a\bar{b}y + \bar{a}b\bar{y} = 0.$$

Hence

$$\begin{aligned} \bar{a}\bar{b} \mathrel{-\!\!<} x \mathrel{-\!\!<} \bar{a} + \bar{b}, \\ \bar{a}b \mathrel{-\!\!<} y \mathrel{-\!\!<} \bar{a} + b. \end{aligned}$$

The equation is therefore soluble, not only without condition, but in a peculiarly simple form.

By an ordinal function I mean a function of four terms which has the form here illustrated by p.[3]

In the foregoing case, p appears as a function of the two unknowns (x, y), and of the two coefficients (a, b), together with their negatives $(\bar{a}, \bar{b})$. For our further purposes, this distinction between the "unknowns" and "the coefficients" may be left out of account. But we still need to note that an ordinal function is to be viewed as a function of four terms which are regarded as first divided into two pairs, while the function itself depends upon the way in which these pairs are chosen, are arranged, and then are submitted to the operations which characterize the form above illustrated.

[3] The ordinal functions here defined form one special case in the class of functions which are sums or products of Whitehead's "secondary primes." But Whitehead does not discuss in any detail this special case.

A few examples may help to make clear what is essential to the form of an ordinal function. Let the four elements (a, b, c, d) and their respective negatives be used. Let

$$\begin{aligned} p &= abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d}, \\ q &= a\bar{b}c + \bar{a}\bar{b}\bar{c} + a\bar{c}d + \bar{a}c\bar{d}, \\ r &= b\bar{c}d + \bar{b}c\bar{d} + ab\bar{d} + \bar{a}\bar{b}d, \\ s &= \bar{a}\bar{c}\bar{d} + acd + ab\bar{c} + \bar{a}\bar{b}c. \end{aligned}$$

In the formation of each of these functions there is followed a set of rules which may be grasped by analyzing first the structure of the function p.

In order to form p we first select the two pairs (a, b) and (c, d). Let the pair (a, b) be called, for the moment, the *first* of these pairs, and the pair (c, d) the *second* pair. Each of these pairs, as it is here written, has its own first and its own second member. p is then defined by means of the five directions which here follow:

1. Take the continued product of both elements of the first pair into the first element of the second pair. We thus get the product abc.

2. Then form the product of the respective negatives of both elements of the first pair into the negative of the first member of the second pair. We now have the product $\bar{a}\bar{b}\bar{c}$.

3. Next form the continued product of the first member of the first pair, into the negative of the second member of the first pair, and into the second member of the second pair. We thus form the product $a\bar{b}d$.

4. Hereupon form the product of the negative of the first member of the first pair, into the second member of the first pair, and into the negative of the second member of the second pair. This gives us the product $\bar{a}b\bar{d}$.

5. Finally, take the sum of the four products thus successively formed. This sum gives us $g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d}$.

It will facilitate a survey of the rule whereby an ordinal function is formed, to use the following new symbolism. Let

$$abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d} = (ab\,;\, cd).$$

Here the expression which follows the symbol $=$ is a new symbol introduced, quite arbitrarily, simply to remind us of the foregoing five directions. This new symbol has the advantage of showing that the pairs (a, b) and (c, d) are distinguished, for the purpose here in question, as the first and the second pair. In the arbitrary symbol $(ab\,;\, cd)$, the semicolon separates these pairs. And the expressions ab, and cd, when thus separated by a semicolon, and together inserted, in this order, in a parenthesis, are not, in that context, to be understood as symbolizing logical products. The expression: $g = (ab;\ cd)$ is to be read: "The function g is that function of the four elements (a, b, c, d), which is formed by taking (a, b) and (c, d) as distinct pairs, whereof (a, b) is the first pair and (c, d) the second, and by then applying the five directions given above. Since these directions are themselves quite general, it would be easy to interpret as determinate ordinal functions any analogous expression, such as $x = (pq;\ rs)$; $y = (pr;.\ qs)$, etc. Thus $x = (pq;\ rs)$ is equivalent, in terms of the ordinary Boolean calculus, to the equation:

$$x = pqr + \bar{p}\bar{q}\bar{r} + p\bar{q}s + \bar{p}q\bar{s}.$$

And $y = (pr;\ qs)$ is equivalent to the equation:—

$$\begin{aligned} y &= prq + \bar{p}\bar{r}\bar{q} + p\bar{r}s + \bar{p}r\bar{s} \\ &= pqr + \bar{p}\bar{q}\bar{r} + p\bar{r}s + \bar{p}r\bar{s}. \end{aligned}$$

As to the following functions of the tetrad (a, b, c, d):

$$\begin{aligned} h &= abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc, \\ m &= abc + \bar{a}b\bar{c} + a\bar{c}d + \bar{a}c\bar{d}, \end{aligned}$$

it is plain that they may be written:

$$\begin{aligned} h &= (ab;\ d\bar{c}) = (ab;\ \bar{c}d), \\ m &= (ac;\ \bar{b}d) = (a\bar{c};\ d\bar{b}). \end{aligned}$$

For if we choose any one of the four expressions here written in parentheses, and if we regard each expression as a shorthand direction to apply our rule for forming ordinals to the two pairs which the semicolon (as used in each of these cases), separates, the resulting functions are as we have just written them.

In sum then, I mean, by an ordinal function, a function of four elements such that, if we begin with the form $(ab;\ cd)$, interpreted as above, we can form all possible ordinal functions by substituting for the tetrad (a, b, c, d) other tetrads of symbols which stand for logical elements; by changing the order of the elements and of the pairs which are in question; and by introducing the negatives, $\bar{a}$, $\bar{b}$, etc., into the expressions.

In case we confine ourselves to the elements (a, b, c, d) and their respective negatives, the permutations and arrangements possible in defining the forms $(ab;\ cd)$, $(\bar{a}\bar{b};\ \bar{d}c)$ $(ca;\ db)$ $(a\bar{d};\ \bar{b}c)$, etc., appear, at first sight, too numerous for an easy survey. But there are relations between pairs of formally distinct ordinal functions which greatly simplify the task of following the variations that our definition permits. Of these relations, the most important is expressed by the symbolic equation:

$$(ab;\ cd) = (cd;\ ab).$$

In the ordinary notation this becomes:

$$abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d} = acd + \bar{a}\bar{c}\bar{d} + \bar{b}cd + \bar{b}\bar{c}d.$$

The verification of this latter equation requires only a very simple computation. But the property expressed is especially characteristic of the ordinal functions. In view of the solution above given in case of an equation whose left-hand member is an ordinal function, it may require but a little reflection to see that the transitivity of the illative relation, the theory of elimination, as it forms part of the theory of logical equations, and consequently the ordinary theory of the syllogism, may all of them be viewed as standing in a close relation to this fundamental principle of the theory of the ordinal functions.

In consequence of the foregoing, we may now readily verify the symbolic equations

$$(ab\,;\,cd) = (cd\,;\,ab) = (a\bar{b}\,;\,dc) = (\bar{a}\bar{b}\,;\,\bar{c}\bar{d}) = (c\bar{d}\,;\,ba) = (\bar{c}\bar{d}\,;\,\bar{a}\bar{b}).$$

The negation of ordinal functions leads to interesting forms.

If

$$g = (ab\,;\,cd),\ \text{then}\ \bar{g} = (ab\,;\,\bar{c}\bar{d}) = (cd\,;\,\bar{a}b).$$

For

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d};$$

and therefore,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd;$$

while, by the foregoing,

$$g = acd + \bar{a}\bar{c}\bar{d} + bc\bar{d} + \bar{b}\bar{c}d;$$

and therefore,

$$\bar{g} = \bar{a}cd + a\bar{c}\bar{d} + \bar{b}c\bar{d} + b\bar{c}d.$$

When we study the ordinal functions which may be derived from $(ab\,;\,cd)$ by changing the order of the elements or of the pairs, and by introducing, in various possible ways, the symbol for negation, we find cases where two such functions are not, in general, equivalent, but become so upon condition that the elements of the tetrad (a, b, c, d) stand in some definite relation to one another. Thus let $g = (ab\,;\,cd)$, while $h = (ba\,;\,dc)$.

Then

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d},$$
$$h = abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc = (ab\,;\,d\bar{c}).$$

Hence,

$$\bar{g} = ab\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{d} + \bar{a}bd = (ab\,;\,\bar{c}\bar{d}),$$
$$\bar{h} = ab\bar{d} + \bar{a}\bar{b}d + a\bar{b}c + \bar{a}b\bar{c} = (ab\,;\,\bar{d}c).$$

In consequence,

$$g\bar{h} + \bar{g}h = abc\bar{d} + ab\bar{c}d + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + a\bar{b}cd + a\bar{b}\bar{c}\bar{d} + \bar{a}bcd + \bar{a}b\bar{c}\bar{d}.$$

Or, more briefly stated,

$$g\bar{h} + \bar{g}h = (ab + \bar{a}\bar{b})(c\bar{d} + \bar{c}d) + (a\bar{b} + \bar{a}b)(cd + \bar{c}\bar{d}).$$

If $g = h$, then both the first and the second members of this equation reduce to zero. In that case we have: $a\bar{b} + \bar{a}b = cd + \bar{c}d$. This last expression is equivalent to asserting $T(abcd)$.

Therefore, if we transform the ordinal $g = (ab; cd)$ into the ordinal $(ba; dc) = h$, by separately reversing the order of the members of the first and the second pairs of the symbol for g, while leaving all else unchanged, then the ordinals g and h can be mutually equivalent if, and only if, $T(abcd)$. The T-relation thus stands in close connection with the properties of the ordinal functions.

These few cases serve to give a first glimpse of the decidedly interesting properties of the ordinal functions. I omit further developments at this stage of the inquiry; but I hope to have more to say about the ordinals in a subsequent paper.

V

Directed Pairs, and an Operation upon Such Pairs

We are now ready to define an operation which is based upon the operations of the ordinary algebra of logic, but which is applied to a new system of logical entities.

These new entities are *pairs* of Boolean logical elements. Just as, in the familiar modern theory of the rational numbers, a rational number is defined as a pair of whole numbers, so, in the theory to which I now pass, the entities to be considered are not Boolean elements, such as a, b, c, etc., but are pairs of such elements. And precisely as, in the modern theory of the rational numbers, the pairs of natural numbers which are in question are *directed* pairs, each pair having its first and its second member, or its upper and its lower member, so, in the following theory of logical pairs, I shall deal with directed logical pairs. Let the symbol a/b, or $\frac{a}{b}$, be used for the directed pair consisting of the elements (a, b), the element a being taken as the first, and b as the second member.

Our foregoing sketch of the properties of the ordinals has called our attention to the pairs of elements which are used in defining such an ordinal as $g = (ab; cd)$. But the pairs of which we made mention in our foregoing account of the ordinals, were not treated *as* pairs. The elements used entered, in a determinate way, into the definition of g. But g was itself a Boolean element. The functions in questions were Boolean functions. The elements a, b, c, etc., were treated only in so far as they were subjected to the operations of the Boolean calculus. The symbol $(ab; cd)$ was itself a mere shorthand for expressing the rule whereby the formation of an ordinal function was guided.

But from this point onwards we are to deal with our directed pairs as new entities, and are to subject them to new operations.

VI

The Ordinal Pair-Operation

Let a/b, c/d be the symbols, respectively, of the two directed pairs (a, b), (c, d). The form a/b means simply that a is taken as the first, and b as the second member of the pair (a, b), for the purpose of the operations into which these pairs are to enter. The symbol c/d has an analogous interpretation in case of the pair (c, d). This adoption of the symbols which are ordinarily used for fractions has in this case *merely* the significance which is given to it by the definition just stated.

Now the two directed pairs a/b and c/d may be so combined as to define, uniquely, a new pair. This pair shall be defined as the directed pair of ordinals g/h, when $g=(ab\,;\,cd)$ and $h=(ba;\,dc)$. Let the combination in question be viewed as an operation upon the pairs a/b, c/d. Symbolize this operation by $\circ$. Then

$$\frac{a}{b}\circ\frac{c}{d}=\frac{g}{h}=\frac{(ab\,;cd)}{(ba\,;dc)}.$$

It is plain that the directed pair g/h is uniquely determined by the directed pairs a/b, c/d.

Since $(ab;\,cd)=(cd;\,ab)=g$, while $(ba;\,dc)=(dc;\,ba)=h$, the operation $\circ$, as now defined, is commutative. That is, $a/b\circ c/d=c/d\circ a/b$.

Hereupon, we come to that feature of the new operation which constitutes the first contrast by which it is distinguished both from the addition and from the multiplication of the ordinary Boolean calculus. The new operation, namely, is *invertible*. That is, given the pair g/h, and either of the pairs a/b or c/d, the other of these pairs is uniquely determined.

For from the equations

$$\begin{aligned} g &= abc+\bar a\bar b\bar c+a\bar bd+\bar ab\bar d \\ &= acd+\bar a\bar c\bar d+bc\bar d+\bar b\bar cd, \\ h &= abd+\bar a\bar b\bar d+a\bar b\bar c+\bar abc \\ &= bcd+\bar b\bar c\bar d+\bar ac\bar d+a\bar cd \end{aligned}$$

we can deduce, by an easy computation, the consequences:—

$$\begin{aligned} a &= cdg+\bar c\bar d\bar g+c\bar d\bar h+\bar cdh=(cd;\,g\bar h)=(dc;\,gh), \\ b &= cdh+\bar c\bar d\bar h+c\bar dg+\bar cd\bar g=(cd;\,hg)=(hg;\,cd), \\ c &= abg+\bar a\bar b\bar g+a\bar b\bar h+\bar abh=(ab\,;\,g\bar h)=(ba;\,gh), \\ d &= abh+\bar a\bar b\bar h+a\bar bg+\bar ab\bar g=(ab\,;\,hg)=(hg;\,ab). \end{aligned}$$

The interest of these facts is greatly increased by the consideration that the invertibility of our operation is possible universally and without restriction, and can be expressed in a peculiarly simple form. Thus, if

$$a/b \circ c/d = g/h,$$

it is easy to show that

$$g/h \circ b/a = c/d,$$

while

$$g/h \circ d/c = a/b.$$

This, in fact, is precisely what is formulated by means of the four equations written above, and expressive of the values of a, b, c, and d.

The novelty and simplicity of the considerations here involved, make it worth while to express the facts in the form of what may be called a "Newlin-diagram,"[4] which brings before the eye the divisions of a universe of discourse containing four independent terms. It is well to remember, however, that any such diagram expresses a special application of the Boolean calculus, while the laws here in question hold true of the pure algebra, and are independent of all applications to "classes," to "areas," or to other special sorts of entities.

		a		$\bar{a}$		
		$a\bar{b}$	ab	$\bar{a}b$	$\bar{a}\bar{b}$	
c	$c\bar{d}$	$\bar{g}\bar{h}$	$g\bar{h}$	gh	$\bar{g}h$	
	cd	$g\bar{h}$	gh	$\bar{g}h$	$\bar{g}\bar{h}$	d
$\bar{c}$	$\bar{c}d$	gh	$\bar{g}h$	$\bar{g}\bar{h}$	$g\bar{h}$	
	$\bar{c}\bar{d}$	$\bar{g}h$	$\bar{g}\bar{h}$	$g\bar{h}$	gh	
			b			

Let

$$g = abc + \bar{a}\bar{b}\bar{c} + a\bar{b}d + \bar{a}b\bar{d} = (ab\,;\, cd),$$
$$h = abd + \bar{a}\bar{b}\bar{d} + a\bar{b}\bar{c} + \bar{a}bc = (ba\,;\, dc).$$

The inverse operation just noted can easily be read off from this diagram. The formal analogy of our new operation, and of its inverse, to the multiplication and division of ordinary rational numbers, is, as far as it goes, worthy of notice. In case of the rational

[4] See the article by Professor W. J. Newlin, in this JOURNAL, Vol. III., page 539.

numbers, to divide by a rational number is equivalent to multiplying by the inverse, that is, by the reciprocal of the pair of whole numbers which constitutes the divisor. Our pairs of Boolean entities obey, with respect to our new operation, an analogous rule. The inverse of our operation is the direct operation with one of the factors of the original operation inverted, and then combined with the result of the original operation.

VII

The Associative Property of the Ordinal Pair Operation

We next come to a still more important fact. Our new operation is not only commutative and invertible, but also associative. That is, if we have given to us the three independent pairs a/b, c/d, e/f, then

$$\left[\frac{a}{b}\circ\frac{c}{d}\right]\circ\frac{e}{f}=\frac{a}{b}\circ\left[\frac{c}{d}\circ\frac{e}{f}\right]$$

The computations needed to establish this property of our operation are of necessity a little diffuse for complete statement in so summary a paper as the present one. It may be worth while, however, to give an outline of what seems to be a convenient mode of dealing with the matter, leaving to the reader the verification of the details of the computation, if he chooses to work them out. It may be remarked that, by means of a six-term "Newlin-diagram," or better by means of a pair of such diagrams, each diagram presenting to the eye one of the associations of pairs which is in question, the associative property of our operation can be made visible. But a six-term Newlin-diagram is somewhat troublesome to print.

It is worth noting, and is easily verifiable, that, if $a/b \circ c/d = g/h$, then

$$\begin{aligned}
gh &= abcd + \bar{a}bc\bar{d} + a\bar{b}\bar{c}d + \bar{a}\bar{b}\bar{c}\bar{d} = (ad+\bar{a}\bar{d})(bc+\bar{b}\bar{c}),\\
g\bar{h} &= abc\bar{d} + a\bar{b}cd + \bar{a}b\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}d = (ac+\bar{a}\bar{c})(b\bar{d}+\bar{b}d),\\
\bar{g}h &= a\bar{b}\bar{c}\bar{d} + abcd + abcd + \bar{a}\bar{b}c\bar{d} = (a\bar{c}+\bar{a}c)(bd+\bar{b}\bar{d}),\\
\bar{g}\bar{h} &= a\bar{b}c\bar{d} + ab\bar{c}\bar{d} + \bar{a}b\bar{c}d + \bar{a}\bar{b}cd = (b\bar{c}+\bar{b}c)(a\bar{d}+\bar{a}d).
\end{aligned}$$

All this may be read off directly from the Newlin diagram printed above. Hereupon we may set $g/h \circ e/f = r/s$.

That is,

$$\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{e}{f}=\frac{r}{s}.$$

Now, by the definition of our operation,

$$\frac{g}{h}\circ\frac{e}{f}=\frac{r}{s}=\frac{ghe+\bar{g}\bar{h}\bar{e}+g\bar{h}f+\bar{g}h\bar{f}}{ghf+\bar{g}\bar{h}\bar{f}+g\bar{h}\bar{e}+\bar{g}he}.$$

Substituting, in the right-hand member of this equation, the values of gh, $g\bar{h}$, $\bar{g}h$, and $\bar{g}\bar{h}$, as given above, we find:

$$r = (ad + \bar{a}\bar{d})(bc + \bar{b}\bar{c})e + (a\bar{d} + \bar{a}d)(b\bar{c} + \bar{b}c)\bar{e}$$
$$+ (ac + \bar{a}\bar{c})(b\bar{d} + \bar{b}d)f + (a\bar{c} + \bar{a}c)(bd + \bar{b}\bar{d})\bar{f},$$
$$s = (ad + \bar{a}\bar{d})(bc + \bar{b}\bar{c})f + (a\bar{d} + \bar{a}d)(b\bar{c} + \bar{b}c)\bar{f}$$
$$+ (a\bar{c} + \bar{a}c)(bd + \bar{b}\bar{d})e + (ac + \bar{a}\bar{c})(b\bar{d} + \bar{b}d)\bar{e}.$$

But the directed pair r/s is, as we have defined it, the equivalent of

$$\left(\frac{a}{b} \circ \frac{c}{d}\right) \circ \frac{e}{f}.$$

Hereupon let us set

$$\frac{c}{d} \circ \frac{e}{f} = \frac{m}{n},$$

and suppose

$$\frac{a}{b} \circ \frac{m}{n} = \frac{u}{v}.$$

Then, by our definition,

$$\frac{a}{b} \circ \left(\frac{c}{d} \circ \frac{e}{f}\right) = \frac{u}{v}.$$

We have also

$$\frac{m}{n} = \frac{(cd\,;\,ef)}{(dc\,;\,fe)} = \frac{cde + \bar{c}\bar{d}\bar{e} + c\bar{d}f + \bar{c}d\bar{f}}{cdf + \bar{c}\bar{d}\bar{f} + c\bar{d}\bar{e} + \bar{c}de}.$$

Furthermore

$$\frac{a}{b} \circ \frac{m}{n} = \frac{u}{v}.$$

Hence

$$u = abm + \bar{a}\bar{b}\bar{m} + a\bar{b}n + \bar{a}b\bar{n}.$$

Substituting the values of m and of n, respectively, as defined above, we discover hereupon that

$$u = ab(cde + \bar{c}\bar{d}\bar{e} + c\bar{d}f + \bar{c}d\bar{f})$$
$$+ a\bar{b}(cdf + \bar{c}\bar{d}\bar{f} + c\bar{d}\bar{e} + \bar{c}de)$$
$$+ \bar{a}b(cd\bar{f} + \bar{c}\bar{d}f + c\bar{d}e + \bar{c}d\bar{e})$$
$$+ \bar{a}\bar{b}(cd\bar{e} + \bar{c}\bar{d}e + c\bar{d}\bar{f} + \bar{c}df).$$

Rearranging this expression for the value of u, we find:

$$u = e(ad + \bar{a}\bar{d})(bc + \bar{b}\bar{c}) + \bar{e}(a\bar{d} + \bar{a}d)(b\bar{c} + \bar{b}c)$$
$$+ f(ac + \bar{a}\bar{c})(b\bar{d} + \bar{b}d) + \bar{f}(a\bar{c} + \bar{a}c)(bd + b\bar{d}).$$

Comparing this result with the value above given for r, we find that $r = u$.

By a precisely analogous computation we find that $s = v$, and thus we reach the result that:

$$\left(\frac{a}{b} \circ \frac{c}{d}\right) \circ \frac{e}{f} = \frac{a}{b} \circ \left(\frac{c}{d} \circ \frac{e}{f}\right) = \frac{r}{s} = \frac{u}{v}.$$

Our operation is therefore not only invertible and commutative, but also associative. Accordingly, with respect to our new operation, the directed pairs of Boolean entities constitute an Abelian group.

VIII

Results

The properties of this group of Boolean pairs are sufficiently remarkable to deserve a summary statement. Each of the following propositions can easily be verified. The computations involved are, on the basis of the foregoing, all of them exteremely simple. They are also, so far as I know, novel.

The system of pairs now defined possesses a unit. This unit is the pair 1/1. For, by definition,

$$\frac{a}{b} \circ \frac{1}{1} = \frac{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}1 + \bar{a}b\bar{1}}{ab1 + \bar{a}\bar{b}\bar{1} + a\bar{b}\bar{1} + \bar{a}b1} = \frac{ab + a\bar{b}}{ab + \bar{a}b} = \frac{a}{b}.$$

That is, whatever pair a/b be combined with the pair 1/1, is left invariant by that combination under the rules of our operation.

Owing to the formal analogy between our ordinal pair-operation and the multiplication of rational numbers (that analogy, namely, of which we made mention above, in speaking of the inverse of our operation), we may treat our ordinal pair-operation as a multiplication, although I have, at present, no addition-operation to set side by side with it. Regarding our operation, then, as a multiplication, we may write:

$$\frac{a}{b} \circ \frac{a}{b} = \left(\frac{a}{b}\right)^2.$$

We may use similar expressions for higher powers of a/b, and speak of cubes, etc. Only here we quickly find ourselves limited by the interesting further group-properties of our pair-operation which may next be stated.

We have, namely:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \circ \frac{a}{b} = \frac{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}b + \bar{a}b\bar{b}}{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}\bar{a} + \bar{a}ba}.$$

Hence

$$\left(\frac{a}{b}\right)^2 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}}.$$

The square of any pair is, therefore, a pair consisting of equal terms. Each of these equals is a "prime-function," *viz.*, the function, $ab + \bar{a}\bar{b}$, of the members of the pair.

We therefore have, for the cube of any pair, the expression:

$$\left(\frac{a}{b}\right)^3 = \frac{ab + \bar{a}\bar{b}}{ab + \bar{a}\bar{b}} \circ \frac{a}{b} = \frac{(ab + \bar{a}\bar{b})a + (a\bar{b} + \bar{a}b)\bar{a}}{(ab + \bar{a}\bar{b})b + (a\bar{b} + \bar{a}b)\bar{b}} = \frac{ab + \bar{a}b}{ab + a\bar{b}} = \frac{b}{a}.$$

That is to say, the cube of any pair is the inverse of that pair. Or, if the inverse of our "multiplication" be regarded, for the present purpose, as a "division," we now observe that to "multiply" by the cube of any pair is equivalent to multiplying by the inverse of that pair, or is, in other words, equivalent to dividing by that pair. In still another expression our result is that, if $a/b \circ c/d = g/h$, then $(a/b)^3 \circ g/h = c/d$; while $g/h \circ (c/d)^3 = a/b$. We next proceed to the fourth powers of pairs. We have:

$$\left(\frac{a}{b}\right)^4 = \frac{a}{b} \circ \left(\frac{a}{b}\right)^3 = \frac{a}{b} \circ \frac{b}{a} = \frac{1}{1}.$$

The last of these equations is reached as follows:

$$\frac{a}{b} \circ \frac{b}{a} = \frac{abb + \bar{a}\bar{b}\bar{b} + a\bar{b}a + \bar{a}b\bar{a}}{aba + \bar{a}\bar{b}\bar{a} + a\bar{b}\bar{b} + \bar{a}bb} = \frac{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}{ab + \bar{a}\bar{b} + a\bar{b} + \bar{a}b}.$$

Hereupon, we observe that

$$\left(\frac{a}{b}\right)^5 = \left(\frac{a}{b}\right)^4 \circ \frac{a}{b} = \frac{a}{b} \circ \frac{1}{1} = \frac{a}{b}.$$

Thus the fourth power of every pair is the unit pair, while the fifth power of each pair is identical with the pair itself. The "period" or "order" of our pair-operation is five. If we conceive the system of directed Boolean pairs as transformed within itself by combining each pair with itself by means of the pair-operation, and by then passing to the higher powers, the first such transformation substitutes for each pair its square, a determinate pair of equals, whose members are each of them a determinate "prime-function" of the original pair. Next, the "cubes," which are the inverses of the original pairs, are produced. The next such transformation substitutes for each and all the pairs the unit pair. Combining this unit pair with each member of the original system leaves that system invariant.

IX

Modulus Pairs

The pair 1/1, the unit pair, may be called the modulus of our ordinal pair-operation. It is evident that, whatever element a may be, the equation $(a/a)^2 = 1/1$ is always true. That is, the modulus is the square of any pair that consists of equal Boolean elements.

Our system of directed pairs contains, however, other pairs which have the properties of moduli; for each such modulus pair may be regarded as the unit of an ordinal pair-operation whose group is the same as the one which we have just been studying, and whose properties are precisely analogous to those of the operation which we have been studying, so that all these operations are variations of a single one. The situation is briefly to be summed up as follows:

Let us consider the four pairs 1/1, 0/0, 1/0, 0/1, in their relations to one another, and to the other pairs of our system.

It is easy to show that

(a) $$\left(\frac{0}{1}\right)^2 = \frac{0}{0},$$

(b) $$\left(\frac{1}{0}\right)^2 = \frac{0}{0},$$

(c) $$\left(\frac{0}{0}\right)^2 = \frac{1}{1},$$

(d) $$\left(\frac{1}{1}\right)^2 = \frac{1}{1},$$

(e) $$\frac{1}{1} \circ \frac{0}{0} = \frac{0}{0},$$

(f) $$\frac{1}{0} \circ \frac{0}{1} = \frac{1}{1},$$

(g) $$\left(\frac{1}{0}\right)^3 = \frac{0}{1}.$$

And thus the mutual relations of the four modulus elements are stated.

But when we combine a pair a/b with each of the four moduli in succession, we get the following results:

(1) $$\frac{a}{b} \circ \frac{1}{1} = \frac{a}{b} \text{ whose cube is } \frac{b}{a},$$

(2) $$\frac{a}{b} \circ \frac{0}{0} = \frac{\bar{a}}{\bar{b}} \text{ whose cube is } \frac{\bar{b}}{\bar{a}},$$

(3) $$\frac{a}{b} \circ \frac{0}{1} = \frac{\bar{b}}{a} \text{ whose cube is } \frac{a}{\bar{b}},$$

(4) $$\frac{a}{b} \circ \frac{1}{0} = \frac{b}{\bar{a}} \text{ whose cube is } \frac{\bar{a}}{b}.$$

If, hereupon, we ask what directed pairs can be formed from a given pair a/b, by considering the four Boolean elements a, $\bar{a}$, b, $\bar{b}$, and by treating their various pairs as directed pairs, we see that the foregoing table of eight directed pairs contains all of the possible combinations, and shows how all the eight can be formed from any one of their number by using the two operations of applying the four moduli, and of raising to the third power.

But the processes in question can be greatly simplified by considering that all the four moduli can be derived from a single one of their number, by merely using our ordinal pair operation. The modulus chosen for this purpose may be either 0/1 or 1/0, at pleasure. Thus, if we begin with 0/1, we derive the other moduli simply by considering the powers of 0/1. For we have:

$$\left(\frac{0}{1}\right)^2=\frac{0}{0};\quad \left(\frac{0}{1}\right)^3=\frac{1}{0};\quad \left(\frac{0}{1}\right)^4=\frac{1}{1}.$$

Starting with any pair a/b, and with the single modulus 0/1, we can therefore form all the derivative pairs $\bar{a}/b$, $a/\bar{b}$, etc., merely by repeating the processes of combining with the modulus, and of raising to powers.

It is possible, however, to define a new operation such that one of the moduli, say 0/1, is the unit pair of this operation. The latter will then be derived from (and in essence equivalent to) our present ordinal pair-operation. Let us use $\cup$ as the symbol of the new operation whereof 0/1 is to be the unit pair. That is, let us require an ordinal pair-operation $\cup$ to be defined such that $a/b \cup 0/1 = a/b$ whatever pair a/b may be. To this end we have only to define $\cup$ by the equation

$$\frac{a}{b}\cup\frac{c}{d}=\left(\frac{a}{b}\circ\frac{c}{d}\right)\circ\frac{1}{0}.$$

For then

$$\frac{a}{b}\cup\frac{0}{1}=\frac{a}{b}\circ\frac{0}{1}\circ\frac{1}{0}=\frac{a}{b}\circ\frac{1}{1}=\frac{a}{b}.$$

The new operation will be so related to the old that, if $a/b \circ c/d = g/h$, then, by definition, $a/b \cup c/d = g/h \circ 1/0 = h/\bar{g}$.

It is plain that by the use of the modulus element, and by raising to powers, all the results of the new operation $\cup$ can be stated in terms of our foregoing operation $\circ$, and conversely. The only novelty of the operation $\cup$ will, therefore, depend upon its choice of one of the modulus elements as its special unit.

The four moduli of our system of directed pairs are themselves pairs, and are not ordinary Boolean elements. They serve to give to the whole system properties that I believe to be not only of interest in themselves, but of no small promise for the future. In any case, here is a definite extension of the Boolean calculus, and a definite and new introduction of group-theory into this realm of the algebra of logic.

Chapter XVI

THE PRINCIPLES OF LOGIC

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§ I. A VERY frequent account of the office of Logic runs substantially as follows: "Logic is a Normative Science. It deals, namely, with the Norms whereby sound or correct thinking is distinguished from incorrect thinking. It consists of two parts,—a general part, called Formal Logic, which defines the universal or formal normative principles to which all correct thinking must conform, and a special and very extended part called Applied Logic, or Methodology, which deals with the norms of thought in their application to the methods used in various special sciences."

From this conventional account the present sketch will deliberately depart. A discussion of some of the more important problems of Methodology will be comprised in our first section. The remaining paragraphs of this paper will be devoted to indicating, very summarily, the nature of a doctrine of which the traditional General or Formal Logic is but a part, and, in fact, a very subordinate part. To this doctrine the name "The Science of Order" may be given. It is a science which is indeed incidentally concerned with the norms of the thinking process. But its character as a normative doctrine is wholly subordinate to other features which make it of the most fundamental importance for philosophy. It is to-day in a very progressive condition. It is in some notable respects new. It offers inexhaustible opportunities for future progress.

§ 2. Everyone will agree that throughout its history Logic has been concerned with the conduct and with the results of the thinking process. Now the thinking process is indeed, from its very nature, *methodical.* In every human science, and in every human art that is teachable at all, the thinking process appears either as the creator and the guide, or else as the formulator and the analyzer, of the methods which characterize this science or art. If an art grows up instinctively, as the product of social need and of individual talent, the efforts to teach this art, so that it may pass from master to apprentice, lead sooner or later to an analysis and thoughtful formulation of the methods employed by the skilful workman. And when an art or a science is deliberately invented or advanced by the conscious skill of the individual inquirer or discoverer, the procedure used either includes a purposeful application of already known methods to new undertakings, or else involves an effort to create new methods. Everywhere, then, the consciousness of method grows in proportion as thought comes to play a successful part in the organization of human life.

Since, however, the methods used vary with the different arts and sciences, and yet have certain important features that are common to all or to many of the undertakings of these arts and sciences, it is natural that a comparative study of methods should form the topic of a more or less independent body of doctrine. And, as a fact, such a Methodology, such a "Normative doctrine," such an effort to survey and to systematize the methods used by all, or by one or another great body of thoughtful workers, has repeatedly constituted the principal task assigned to Logic, whether the distinction between General or Formal Logic and Applied Logic has been emphasized or not. Logic as a branch of philosophy began, as is well known, when the differences of opinion amongst the various philosophers, when the dialectical problems brought to notice by the Eleatic school, and when the more or less practical inquiries of the Sophists into the arts of disputation and of persuasion, had led to a conscious need for a general study of the methods of right thinking. In Aristotle's case the task of surveying, and in part of creating, a systematic body of sciences constituted an additional ground for undertaking a general methodology of the thinking process. And ever since Aristotle the view that one main purpose of Logic is to expound the "Art of Thinking," or the definition of Logic in some other more or less exclusively methodological fashion, has played a large part in the history of our science. And this is why the definition of Logic as a Normative Science is still so common, and in its place useful.

As a fact, however, Methodology, taken in its usual sense as a study of the norms and methods of thought used in the various arts and sciences, is the mother of Logic taken in the other sense hereafter to be expounded. For the undertakings of Methodology lead to certain special problems, such as Plato and Aristotle already began to study, and such as recent inquiry makes more and more manifold and important. These problems, when considered for their own sake, assume an aspect that pretty sharply differentiates them from the problems of Methodology proper. They are problems regarding, *not* the methods by which the thinker succeeds, nor yet the norms of correct thinking viewed as norms, but rather the *Forms*, the *Categories*, the *Types of Order*, which characterize any realm of objects which a thinker has actually succeeded in mastering, or can possibly succeed in mastering, by his methods. Taken in this sense, *Logic is the General Science of Order*, the *Theory of the Forms of any Orderly Realm of Objects*, real or ideal.

Just because Logic, viewed as such a doctrine, has resulted from the efforts to formulate the norms and methods of thinking, the question how Logic as Methodology differs from and yet gives birth to Logic conceived as the Science of Order, must be summarily indicated in the rest of our opening section. To this end, we must consider some of the principal problems of Methodology.

§ 3. Let us then first return to a brief mention of some of the problems of method which characterized the well known early stages of logical inquiry, as they are represented, for instance, in remarks that frequently recur in the Platonic dialogues.

The "plastic youth" of the Platonic dialogues, is to be instructed by Socrates in the right method of thinking, and is to be warned against the false arts of the Sophists. The instruction that he most frequently receives relates: (1) To the proper method of definition; (2) To the task of systematic classification, with the prevailing use of dichotomy for the sake of dividing a wider class into its constituent species; (3) To a careful study of the evidence which attaches to certain notable propositions; (4) To a watchful examination of modes of inference. The special considerations which are so frequently repeated in the Platonic dialogues in regard to each of these matters, do not here, in any detail, concern us. It is enough to recall a few facts only. Definition, for instance, according to the Socratic and Platonic methodology, depends indeed upon a collection of special instances of the concept that is to be defined. But, as Socrates often points out, instances, taken merely as such, constitute no definition. For we do not learn what clay is merely by remembering or by naming several different sorts of clay. One must conceive, in universal terms, what is common to these sorts of clay. And so too it is if we want to define justice, or virtue, or knowledge. Definition gets at the essence, at the "Idea," at the type, which special instances exemplify, and depends upon taking the universal as such, and upon bringing it to our knowledge with clearness. But a definition, once thus formulated upon the basis of the instances first chosen, needs to be further tested. One tests it, according to this methodological doctrine, by applying it to new instances, and by a deliberate search for possible inconsistencies. For a truly universal account of a concept must provide for all the cases that rightfully fall under the concept which is to be defined, and must exclude all instances which do not belong to the type in question. In case inconsistencies are discovered, by finding that the definition includes too much or too little, the definition first attempted must be amended. But in such consideration of right definitions, one is greatly aided by remembering that no universal types exist in isolation. And here a very important feature of Plato's methodology appears. *The universals, the "Ideas," form a system.* There are the more and the less inclusive universals. Instances, or classes of instances, which appear to possess mutually inconsistent characters, may still be conceived as members of the same larger class, and in so far as illustrating the same universal, if only they can be shown to be determined to be thus distinct through a process of classification, whereby the essence of the more inclusive universal is in fact more clearly portrayed than it could be through a merely abstract definition. One knows number, in its universal essence, all the better, when one learns to classify the numbers as even and odd, as perfect squares or as not perfect squares, and so on. Such classifications are very commonly best made in the form of dichotomies.

The class A may be divided into the A that is *b*, and the A that is not *b*. Arrays of classes and sub-classes may be arranged by repeating such a process. And then a sub-class whose traits are very highly specific, may be defined in universal terms by considering, first A (some "highest genus," as, in terms of the later logic, we may already name it); then B, which comprises whatever A possesses the character *b*; then C, which comprises whatever B possesses the differential mark *c*, and so on. Thus definitions may be rendered both consistent and systematic, and the system or true Order of the universals may be at least approached, if not fully grasped.

As for the evidence which attaches to single propositions, that also must be considered in the light of special test-cases, must be subjected to the criterion of consistency, and must be made familiar by repeated examination. In the course of such examination and re-examination of the convictions which most interest the philosopher, the importance of a clear consciousness regarding the nature of correct inference often comes to light. One is clear that one infers rightly, not when one is carried away by the Sophist's torrent of persuasive oratory, but when one observes the necessity of each individual transition from thought to thought. If one believes that "All A is B," a closer examination readily shows the general truth that one may not thence infer that "All B is A." Yet in hasty discourse, or under the influence of a Sophist's oratory, one might let such a false inference pass unheeded.

§ 4. So much may here suffice as a mere hint and reminder of thoughts which now seem methodological commonplaces, but which, at that early stage of the history of Logic, were momentous for the whole future of the subject. The elementary text-books still repeat the substance of these observations, even if their context is no longer that which appears in Plato's dialogues.

It will be noted at once that such a methodology naturally leads to a view of the nature and constitution of the world of truth, whose significance, at least as Plato conceived it, goes far beyond the value of these precepts as guides for the learner of the art of thinking. If, namely, these things are so, then, in Plato's opinion: (1) *The realm of the Universals or "Ideas" is essentially a System*, whose unity and order are of the first importance for the philosopher; (2) *Inference is possible because truths have momentous objective Relations*, definable precisely in so far as the process of inference is definable; (3) *The "Order and Connection" of our rational processes*, when we follow right methods, *is a sort of copy of an order and connection which the individual thinker finds, but does not make.* One thus sets out to formulate the right method. One discovers, through this very effort, a new realm—*a realm of types, of forms, of relations.* All these appear to be at least as real as the facts of the physical world. And in Plato's individual opinion they are far more real than the latter. Thus Methodology leads Plato to a new Ontology. The world of the Forms becomes the world of the Platonic Ideas; and Dialectic, with its methods, becomes for Plato the gateway of Metaphysics. Here he finds the key to unlock the mystery of Being.

We are not in the least concerned to estimate in this discussion the correctness or even the historical significance of the Platonic Metaphysic,—a doctrine thus merely suggested. It is enough to note, however, that even if one sets aside as false or as irrelevant all the principal metaphysical conclusions of Plato, one sees that in any case the Methodology of the logician, even in this early stage of the doctrine, inevitably gives rise to the problem as to the relatively objective order and system of those objects of thought to which the methodologist appeals when he formulates his procedure. The Platonic theory of Ideas, Aristotle's later theory of Forms, the innumerable variations of the Platonic tradition which the subsequent history of thought contains,—all these may or may not be of use in formulating a sound metaphysic. But in any case this comes to light: If a logician can indeed formulate any sound method at all, in any generally valid way, he can do so only because certain objects which he considers when he thinks,—be these objects definitions, classes, types, relations, propositions, inferences, numbers, or other "principles,"—form a more or less orderly system, or group of systems, whose constitution predetermines the methods that he must use when he thinks. This system, or these systems, and their constitution, are in some sense more or less objective. That is: What constitutes order, and what makes orderly method possible, is not the product of the thinker's personal and private caprice. Nor can he "by taking thought" wilfully alter the most essential facts and relations upon which his methods depend. If an orderly classification of a general class of objects is possible, then, however subjective the choice of one's principles of classification may be, there is *something* about the general nature of any such order and system of genera and of species,—something which is the same for all thinkers, and which outlasts private caprices and changing selections of objects and of modes of classification.

Meanwhile (as we may here add by way of general comment), orderliness and system are much the same in their most general characters, whether they appear in a Platonic dialogue, or in a modern text-book of botany, or in the commercial conduct of a business firm, or in the arrangement and discipline of an army, or in a legal code, or in a work of art, or even in a dance or in the planning of a dinner. Order is order. System is system. Amidst all the variations of systems and of orders, certain general types and characteristic relations can be traced. If then the methodologist attempts to conduct thinking processes in any orderly way, he inevitably depends upon finding in the objects about which he thinks those features, relations, orderly characters, upon which the very possibility of definite methods depends. Whatever one's metaphysic may be, one must therefore recognize that there is something objective about the Order both of our thoughts, and of the things concerning which we think; and one must admit that every successful Methodology depends upon grasping and following some of the traits of this orderly constitution of a realm that is certainly a realm of facts.

§ 5. This brief reference to the consequences to which the Socratic and Platonic Methodology so early led, may suffice to suggest a deep connection between Methodology proper, and what we have called the Science of Order. This connection becomes only the more impressive if we pass from those elementary and now commonplace considerations which play their part in the methodological passages of the Platonic dialogues, to a few observations that a brief review of contemporary scientific thinking will readily bring to the mind of any fairly well informed student.

Let us then at once turn from the earliest stages of Logic to its latest phases. Let us here omit any attempt to expound the Aristotelian Logic, or to estimate its methodological value, or to tell its later history. Let us pass over the often repeated story of the Baconian reform of scientific methods and of the vastly more important consequences of the experimental methods which Galileo and his contemporaries introduced into modern science. Let us come directly to the present day; let us remind ourselves of some of the most familiar of the doctrines of modern scientific Methodology; and then let us see how these doctrines also lead us to problems which demand their own special treatment, and which again force us to define a Science of Order,—a science distinct from Methodology proper, but necessary to a true understanding of the latter.

It is a commonplace of modern Methodology that our knowledge of nature is gained through induction, and upon the basis of experience. It is equally a commonplace that scientific induction does not consist merely of the heaping up of the records of the facts of crude experience. Science is never merely knowledge; it is orderly knowledge. It aims at controlling systems of facts. Amongst the vastly numerous methods which various sciences employ in our day, there are some which stand out as especially universal and characteristic means of accomplishing the aim just emphasized. Let us mention the most prominent of these methods. Such mention will at once bring us again into contact with the fundamental problems whose nature we are here attempting to illustrate.

And so, first, every science, in dealing with the facts of experience, employs *Methods of Classification*, and is so far still making its own use of the lessons that Socrates taught. There is, in the development of every new science of nature, a stage in which, in the absence of more advanced insight into the laws to which the facts are subject, classification is the most prominent feature of the science. Botany and Zoology, in their earlier stages of growth, were, for a considerable time, sciences in which classification predominated. Anthropology, in its treatment of the problems presented by the racial distinctions of mankind, is still very largely in the stage of classification; while in other of its fields of work, as, for instance, in its comparative study of the forms and results of human culture, Anthropology now pursues methods which subordinate classification to the higher types of methodical procedure. Amongst the medical sciences, Psychiatry is just emerging from the stage where the classification of cases, of symptoms, and of disorders made up the bulk of the science; and has begun to live upon a higher plane of methods. In the Organic Sciences the stage of classification (as such instances remind us) very generally endures long, and is with difficulty transcended. And the more complex the facts to be understood, the harder it is for any science, organic or inorganic, to get beyond this first stage. In the case of Chemistry we have a notable instance of a science where the complexity of the facts long forced the science to consist in large part of the enumeration and classification of elements, compounds, properties, and reactions, despite the fact that the experimental methods used were especially well adapted to lead to a knowledge of very general and exact laws. Recent Chemistry, however, has grown far beyond the stage of mere classification.

Where a science passes from this early stage to one of higher insight, *two* more or less sharply distinct types of methods, either separately or (as oftener happens) in combination, frequently play a large part in determining the transition. These are (1) The type of the methods that involve *comparing the corresponding stages* in the various *processes or products of natural Evolution* with which the science has to deal; and (2) The Statistical Method proper, that is the method *which uses exact enumerations as the bases of inductions.*

§ 6. In the wholly or partly organic sciences, the Comparative methods just mentioned play a very large part. How they lead, beyond the stage of classification, to higher sorts of knowledge, is well exemplified by the case of Geology. That science began with classifications of rocks and of formations. But almost from the outset of the science it became evident that these formations were not sudden creations, but had been the results of processes that had required long periods of time. The earlier efforts of "Vulcanists" and "Plutonists" to furnish adequate universal theories of these processes in more or less simple terms, showed that other methods must be used. The key to unlock *one* portion of the mysteries which the new science was to explore, was furnished by the comparative study of the geological formations found in various regions of the earth's crust. When this comparison showed, for instance, corresponding series of fossil-bearing strata, a new light was thrown upon the history of the earth. To be sure, such comparative study of geological series of formations and of fossils, constitutes but one portion of the resources of Geology. Other methods, and very different ones, play their part in Dynamical Geology. But the importance of the comparative study of corresponding geological formations for Historical Geology, serves as one example of what makes the comparative method, in its various analogous forms, significant in great numbers of scientific investigations.

Suppose, namely, that what is to be studied consists of the stages or of the results of any evolutionary process whatever. Something has grown, or has resulted from the ageing or from the "weathering" of the crust of a planet, or from the slow accretion of the results of a civilization. Rock formations, or the anatomical constitution of various organisms, or social systems such as those of law, or such as customs, or folklore, or language, are to be understood. One begins with classification. But herewith science is only initiated, not matured. For it is the evolutionary process itself, or the system of such processes, which is to be comprehended. The comparative procedure it is which first *correlates the corresponding stages of many analogous or "homologous" evolutionary processes and products*, and thus enables us not merely to classify but to unify our facts, by seeing how the most various phenomena may turn out to be stages in the expression of some one great process.

§ 7. Side by side with the Comparative Methods stand the Statistical Methods. These two sorts of methods are, in fact, by no means always very sharply to be distinguished. There are various transitions from one to the other. Every comparison of numerous evolutionary processes, or of the results of such processes, involves of course some more or less exact enumeration of the cases compared.

But such enumeration may not be the main object of consideration. Very many statistical enumerations are guided by the definite purpose to carry out with precision the comparative methods just exemplified. But, as the well known applications of statistical methods to insurance, and to other highly practical undertakings show us, the most characteristic features of the statistical procedure are independent of any such interest as leads the geologist to his correlations of corresponding formations, or the comparative philologist to his analysis of corresponding grammatical forms in different related languages. The Statistical Methods are often used as a short road to a knowledge of uniformities of nature whose true basis and deeper laws escape our knowledge. Mortality tables are good guides to the insurance companies, even when medical knowledge of many of the causes of death remains in a very elementary stage. The statistics of marriage and divorce, of suicide and of crime, or of commerce and of industry, furnish bases for sociological research, even when there is no present hope of reducing the science in question to any exact form.

But whatever their uses, the Statistical Methods involve us in certain problems which have to do with the *correlation of series of phenomena*. A glance at any considerable array of statistical results serves to show us how the mere heaping up of enumerations of classes of facts would be almost as useless as the mere collection of disordered facts without any enumeration. Statistical results, in fact, when they are properly treated, serve to describe for us the constitution of objects whose general type Fechner had in mind when he defined his *Collectivgegenstände*. Such a *Collectivgegenstand* is a conceptual object which results when we conceive a great number of individual facts of experience subjected to a process of thought whereof the following stages may here be mentioned:—

(*a*) These individual facts are classified with reference to certain of the features with respect to which they vary. Such features are exemplified by the varying sizes of organisms and of their organs, by the various numbers of members which different interesting parts of the individual objects in question possess, by the extent to which certain recorded observations of a physical quantity differ from another, and so on.

(*b*) This classification of the facts with reference to their variations having been in general accomplished, the Statistical Method enumerates the members of each of the classes, in so far as such enumeration is possible or useful.

(*c*) The various enumerations, once made, are arranged in orderly series, with reference to questions that are to be answered regarding the laws to which the variations in question are subject. Such series, in case they are sufficiently definite and precise in their character, tend to show us *how two or more aspects of the phenomena in question tend to vary together*,—as, for instance, how human mortality varies with age; how the mean temperature of a place on the earth's surface varies with its latitude or with the season of the year; how the size of an organ or an organism varies with conditions that are known to be determined by heredity or by environment; and so on.

(*d*) *Various series*, when once defined with reference to such features, *are correlated with one another*, by means which the

Methodology of the various Statistical Sciences has further to consider.

(*e*) And, as a result of such processes, the statistician comes to deal with "aggregates" or "blocks" of facts which, taken as *units*, so to speak, *of a higher order*, appear as possessing a structure in which laws of nature are exemplified and revealed. Such *ordered aggregates treated as units of a higher order are Collectivgegenstände.*

Now it is obvious that every step of such a methodical procedure presupposes and uses the concepts of *number*, of *series*, and of the *correlation of series*; and that the whole process, when successful, leads to the establishment of *an orderly array of objects of thought*, and to the revelation of the laws of nature through the establishment and the description of this order. *The Concept of Order is thus a fundamental one both for the Comparative and for the Statistical Methods.*

§ 8. Both the Comparative Methods and the Statistical Methods are used, in the more developed sciences that employ them, in as close a relation as possible to a method which, in the most highly developed regions of physical science, tends to supersede them altogether. This Method consists in *The Organized Combination of Theory and Experience.* This combination reaches its highest levels in the best known regions of physical science. Its various stages are familiar, at least in their most general features. But the methodological problems involved are of great complexity, and the effort to understand them leads with peculiar directness to the definition of the task of the general Science of Order. Let us briefly show how this is the case. In order to do so we must call attention to a familiar general problem of method which has so far been omitted from this sketch.

By the Statistical and by the Comparative methods, laws of nature can be discovered, not with any absolute certainty, but only with a certain degree of *probability.* The degree of probability in question depends (1) upon the number of instances that have been empirically observed in applying these methods, and that have been compared, or statistically arrayed, and (2) upon the fairness with which these facts have been chosen. Since every induction has as its basis a finite number of empirical data, and in general a number that is very small in comparison with the whole wealth of the natural facts that are under investigation, any result of the comparative or of the statistical methods is subject to correction as human experience enlarges. A question that has always been prominent in the discussion of the general methodology of the empirical sciences, is the question as to our right *to generalize from a limited set of data*, so as to make assertions about a larger, or about an unlimited set of facts, in which our data are included. By the Comparative Method, one learns that such and such sets or series of facts are thus and thus correlated,—as for instance that the geological strata so far observed in a given region of the earth's surface show signs of having been laid down in a certain order, with these and these conformities and non-conformities, faultings, foldings, and so on. How far and in what sense has one a right, by what has been called "extrapolation," to extend the order-system thus defined to more or less nearly adjacent regions, and to hold that any still unobserved geological features of those other regions will be, in their character and order, of the type that one has already actually observed? Or again, by the Statistical Method, one learns that certain facts enable one to define a *Collectivgegenstand* of a certain type. How far can one rightly "extrapolate," and extend one's statistical curves or other statistical order-types, to regions of fact that have not yet been subject to enumeration? For instance, how far can one make use of mortality tables, framed upon the basis of previous records of death, for the purpose of insuring lives in a population which inevitably differs, in at least some respects, from the population that has already met with its fate, and that has had its deaths recorded in the mortality tables?

The general answer to this question has often been attempted by methodologists, and has usually taken the form of asserting that such "extrapolation" logically depends, either upon the principle, "*That nature is uniform*," or upon the still more general principle: "That *every event*" (or, as one sometimes asserts, "*every individual fact*") "*has its sufficient reason.*" It is commonly supposed, then, that the basis of our right to generalize from a limited set of data to a wider range of natural facts, some of which have not yet been observed, may be stated in either one of two ways:—(I.) "These and these facts have been observed to exemplify a certain order-system. But nature is uniform. That is, nature's various order-systems are all of them such as to exemplify either one invariant type, or a certain number of definable and invariant types. Hence the type of the observed facts can be, with due generalization, extended to the unobserved facts." Or again, using the so-called "Principle of Sufficient Reason," one has often stated the warrant for extrapolation substantially thus: (II.) "The facts observed are such as they are, and conform to their own order-system, not by chance, but for some Sufficient Reason. But a sufficient reason is something that, from its nature, is general, and capable of being formulated as a law of nature. The facts still unobserved will therefore conform to this same order type (will exemplify this same law), *unless there is some sufficient reason why they should not conform to this type.* This reason, if it exists, can also be stated in general terms, as another law of nature, and must in any case be *consistent* with the reason and the law that the observed facts have exemplified. Since law thus universally reigns in the natural world, since all is necessary, and since the observed facts not merely are what they are, but, for sufficient reason, *must be* what they are, we ought to regard the laws in terms of which the observed facts have been formulated as applicable to unobserved facts, unless there is a known and probable reason why they should not so conform. To be sure, our conclusion in any one case of such extrapolation is only probable, because it must be admitted, as a possibility, that there may be a sufficient reason why at least some of the unobserved facts should conform to laws now unknown. But the presumption is in favour of extrapolations unless sufficient reason is known why they should not be attempted."

§ 9. Familiar as such modes of stating the warrant for generalizations and extrapolations are, it requires but little reflection to see that the formulations just stated *leave untouched the most important features of the very problem that they propose to solve.* Let us suppose that one who is, in regard to a given scientific field of investigation, a layman, hears the expert give an account of certain uniformities of the data that have been observed in the field in question. So far, of course, the layman is dependent upon the expert for the correctness of the report. If the question then arises, "What right is there to generalize from these observed uniformities, so as to apply them to unobserved facts that belong to this same general field?" is the layman now able to use a general principle "That Nature is uniform," to decide this matter? No! The layman, if properly critical, usually knows that this latter question is quite as much one for the expert to decide, as it is the expert's business to observe or to estimate the uniformities that have already come under observation in his own realm. In the geological case, for instance, the question whether or no certain special features of formations that have already been explored are likely to be repeated in regions not yet subject to geological study, is itself a question for the geologist. It cannot be settled by any appeal to the supposed general principle of the "Uniformity of Nature." That principle, in its abstract formulation, fails to help us precisely when and where we most need help.

Nature, in fact, is indeed full of uniformities. But what these uniformities are is itself a matter for observation. And only the very sort of experience that assures us of certain observed uniformities, can be our guide whenever we attempt to generalize from the observed uniformities to the unobserved ones. Sometimes the fact that certain uniformities have been observed, gives us very good warrant for expecting them to be repeated, in definite ways, in other regions of experience. Sometimes this is not the case, beyond some very limited range. Thus, the fact that a given man has lived ninety years, gives no presumption, based upon the general "uniformity of nature," that he will continue to live long in future. On the contrary, we are accustomed to say that, just because of "the uniformity of nature," as we now know it, he is likely to die soon; because, at his age, whenever an exceptional man chances to reach it, the general death rate is presumably high in proportion to the number of men of ninety years of age.

It follows that, if one uses the principle of the "Uniformity of Nature" as the basis for his extrapolations and generalizations, he has at once to face the question: "What uniformities are of importance in the field in question?" And to this question the *general principle of uniformity gives no answer.* This answer can only come from an empirical study of the uniformities that each region of nature presents.

Equally useless, in aiding us with reference to any one decision regarding our right to generalization and extrapolation, is the direct application of the "Principle of Sufficient Reason." How can we judge, in advance of experience, whether or no there is a "sufficient reason" why the facts not yet observed in a given field should agree in their order-systems with the facts that have already been observed? Surely, by itself, the abstract "Principle of Sufficient Reason," even if fully granted, only assures us that every fact, and so, of course, every order-system of facts, is what it is by virtue of *some* sufficient reason, which is of course stateable in general terms as some sort of a law. But, the very question at issue is whether the still unobserved facts of any given field of inquiry conform to the *same* laws, and so have the *same* "sufficient reasons," as the thus far observed data. This question can admittedly be answered with certainty only when the now unobserved facts have come to be observed. Till then all remains, at best, only "probable." Now the "Principle of Sufficient Reason" does not by itself state any reason why only a *few* laws, or a *few* sorts of sufficient reasons should with probability be viewed as governing nature. It does not, therefore, of itself establish *any* definable probability why there should not actually be a sufficient reason why the unobserved facts should conform to new laws.

Thus neither the abstract principle of the "Uniformity of Nature" nor the still more abstract principle of "Sufficient Reason," serves to assure us of any definite probability that observed uniformities warrant a given generalization or extrapolation into regions not as yet subjected to observation. The question "What observed uniformities are such as to warrant a probable generalization in a given field?" is a question whose answer depends not upon any general application of either of the foregoing principles. They could both hold true in a world whose facts were such as defied our efforts to find out *what* the uniform types in question were, and *what* sufficient reasons there were for any fact that took place.

§ 10. What consideration is it, then, which makes generalizations and extrapolations, upon the basis of already observed uniformities, probable? To this question the American logician, Mr. Charles S. Peirce, has given the answer that is here to be summarized.[1]

This answer will especially aid us in understanding why the methods of comparison, and the statistical methods, inevitably lead, whenever they succeed, to a stage of science wherein the

[1] See Peirce's article on the Logic of Induction in the "Studies in Logic by Members of the Johns Hopkins University" (1883), and his article on "Uniformity," and several passages in his other contributions to Baldwin's *Dictionary of Psychology and Philosophy.*

method which organically unites Theory and Observation, becomes the paramount method. And hereby we shall also be helped to see why the types of Order whose methodical employment characterizes the highest stages of the natural sciences, are the proper topic of a special science that shall deal with their logical origin and with their forms.

Suppose that there exists any finite set of facts such as are *possible objects of human experience*, that is, suppose that there exists a finite set of facts belonging to what Kant calls the realm of *mögliche Erfahrung*. *One* presupposition regarding these facts we may here make, for the sake of argument, without at this point attempting to criticize that presupposition. It is the simple presupposition that these facts, and so the whole aggregate of them, whatever they are, have *some definite constitution.* That is, according to our presupposition, there are possible assertions to be made about these facts which are *either true or false* of each individual fact in the set in question. And, within some range of possible assertions which we here need not attempt further to define, it may be presupposed that: " Every such assertion, if made about any one of those individual facts, and if so defined as to have a precise meaning, either is true or is not true of that fact." Thus, if our realm of " objects of possible experience " is a realm wherein men may be conceived to be present, and if the term *man* has a precise meaning, then the assertion, made of any object A in that realm, "A is a man," either is true or is not true of A. And if our realm of objects is supposed to be one which consists of black and white balls deposited in an urn, the assertion, "A is a white ball," made about one of the balls in the urn, either is true or is false.

This presupposition of the *determinate constitution* of any set of facts such as are subject to inductive investigation, is by no means a simple, not even a " self-evident " presupposition. This, indeed, we shall later have occasion to see. But this presupposition, as Peirce has shown, is the one *natural and indispensable presupposition in all inductive inquiries.* And it is further Peirce's merit, as an inductive methodologist, to have made explicit a consideration which is implicitly employed by commonsense in the ordinary inductive reasonings used in the market place, or in any other region of our practical life. This consideration is that, *if we once grant the single principle of the determinate constitution of any finite set of facts of possible* *experience, we can draw probable conclusions regarding the constitution of such a set of facts, in case we choose "fair samples" of this collection,* and observe their constitution, and then generalize with due precautions. And in order thus to generalize from the sample to the whole collection, *we do not need any presupposition that the collection of facts which we judge by the samples has a constitution determined by any further principle of "uniformity" than is at once involved in the assertion that the collection sampled has in the sense just illustrated, some determinate constitution.* In other words, given a finite collection of facts which has *any* determinate constitution whatever—be this constitution more or less "uniform," be the "sufficient reason" for this constitution some one law, or any possible aggregate of heterogeneous "reasons" whatever—it remains true that we can, *with probability,* although, of course, *only* with probability, judge the constitution of the whole collection by the constitution of parts which are "fair samples" of that whole, even when the collection is very large and the samples are comparatively small.

That we all of us make inductions, in our daily business, which employ the principle of "fair sampling," is easy to see. Peirce has emphasized the fact that the concept of the "fair sample" is not a concept which requires any special presupposition about the uniform constitution of the collection from which we take our samples. It is possible to judge by samples the probable constitution of otherwise unknown cargoes of wheat or of coal, the general characteristics of soils, of forests, of crowds of people, of ores, of rubbish heaps, of clusters of stars, or of collections of the most varied constitution. A mob or a rubbish heap can be judged by "samples" almost as successfully as an organized army or an orderly array of objects, if only we choose from the large collection that is to be sampled a sufficient number of representative instances. And the commercially useful samples employed when cargoes, or other large collections are to be judged, are frequently surprisingly small in proportion to the size of the whole collection that is to be judged by means of them.

§ 11. The reason why such a procedure gives good results can readily be illustrated. Let us take one of the simplest possible instances. Suppose that a certain collection consists of *four* objects, which we will designate by the letters *a*, *b*, *c*, *d*.

And to make our instance still more concrete, suppose that our collection consists in fact of four wooden blocks, which are marked, respectively, by the letters (*a*, *b*, *c*, *d*). Suppose that these blocks are precisely alike, except that they are painted either *red* or *white*. Let us hereupon suppose that somebody is required to judge how *all* the four blocks are colored, by drawing *two* of them at random from a bag in which they are concealed, and by then forming the hypothesis that, just as the colors *white* and *red* are present in the pair that he draws, precisely so these colors will be present and distributed in the whole set of four. In other words, if he draws two white blocks he shall be required to generalize and say: "All four of the blocks are white." If he draws one white and one red block, he shall be required to say: "Half of the blocks (that is, two of them) are red, and the others white."

Suppose next that, as a fact, the blocks *a* and *b* are red, while the blocks *c* and *d* are white. Let us consider what results of such a process of judging the four objects by a sample composed of two of them, are now, under the agreed conditions, *possible*. Of the four blocks (*a*, *b*, *c*, *d*), there are six pairs:—

(*a*, *b*) (*a*, *c*) (*a*, *d*) (*b*, *c*) (*b*, *d*) (*c*, *d*).

Six different samples, then, could be made from the collection of blocks under the supposed conditions. Of these six possible samples, One, namely, the sample (*a*, *b*) would consist, by hypothesis, of two red blocks. Whoever chanced to draw that sample, so that he was consequently required, by the agreement, to judge the whole set by that pair, would judge erroneously; for he would say: "All the four blocks are red." Whoever chanced to draw the pair (*c*, *d*), would have to say: "All the blocks are white." And he too would be wrong. But whoever drew any one of the *four* samples, (*a*, *c*) (*a*, *d*) (*b*, *c*) (*b*, *d*), would by agreement be obliged to say: "Two of the blocks are red and two are white," since he would be obliged, by the agreement, to judge that the whole collection of four showed the same distribution of white and red as was shown in the pair that he had drawn. Thus, if all the possible pairs were independently drawn by successive judges, each one drawing one of the possible pairs from the bag in which the four blocks were hidden then, under the supposed agreement, *two* of the judges would be wrong, and *four* of them right in their judgments.

This simple case illustrates the principle which Peirce uses in his theory of the inductive procedure. In general, if we choose partial collections from a larger collection, and judge the constitution of the whole collection from that of the parts chosen, fixing our attention upon definable characters present or absent, in the partial collections, we are aided towards probable inferences by the fact that there are *more* possible "samples," or partial collections, that at least approximately *agree* in their constitution with the constitution of the whole, than there are samples that widely disagree. Two of the possible samples in the foregoing simple case disagree, four agree, in the character in question, with the collection which is, by the supposed agreement, to be judged by the samples. That is, the possible ways of successful sampling are in this case twice as numerous as the possible unsuccessful ways.

What holds in this simple case holds in a vastly more impressive way when the collections sampled are large. Only then, to be sure, the probable inferences are, in general, only approximations. Suppose a large collection containing m objects. Suppose that a proportion r per cent. of these objects actually have some character q, while the rest lack this character. Suppose that the whole large collection of m objects is to be judged, with reference to the presence or absence of q, by some comparatively small sample containing n of these objects. The success of the judgment will depend upon how far the sample of n objects that happens to be chosen differs from or agrees with the whole collection, with reference to the proportion r' per cent. of the n objects which possess the character q. Of course it is possible that $r=r'$.

In case of large collections and fairly large samples, this will not often be exactly true. But if we consider *all possible selections* of n objects from the collection of m objects, even if n is a comparatively small number, while m is a very large number, a direct calculation will readily show that decidedly *more* of the *possible* sets of "samples" containing n objects will somewhat closely resemble in their constitution the whole collection in respect of the presence or absence of q than will very widely differ in their constitution from that collection. The matter will here in general be one of approximation, not of exact results. If, once more, r' per cent. represents the proportion of the members of a given sample of n objects that possess the character q, while r per cent. is the proportion of the members of the whole collection that possess this same character q, it is possible to compute the number of *possible* samples consisting of n objects each, in which r' will differ from r by not less than or by not more than a determinate amount, x. The computation will show that, as this amount of difference increases, the number of possible samples in question will rapidly decrease.

In consequence, as Peirce points out, our inductive inferences can generally be stated thus, in so far as they involve the direct processes of sampling collections :—

"A proportion r' per cent. of the P's have the character q.
The P's are a 'fair sample' of the large collection M.
"Hence, *probably and approximately*, a proportion r' per cent. of the large collection M have the character q."

The ground for this probability thus rests, not upon the uniformity of the collection M, but upon the fact that more of the possible "fair samples" agree approximately with the whole than widely disagree therewith.

Now a "fair sample" of the large collection M is a sample concerning which we have *no reason to suppose that it has been chosen otherwise than "at random,"* or in a representative way, from among the objects of the large collection that we judge.

Thus the methodology of inductive generalization, so far as the statistical and the comparative methods are concerned, rests simply upon the principle that the facts which we study have a determinate constitution, to which we can approximate, with probability, by fairly sampling the whole through a selection of parts. From its very nature the procedure in question in all such cases is therefore essentially *tentative*, is subject to correction as comparison and statistical enumeration advance from earlier to later stages, and is productive of approximately accurate results, and, in general, of approximations only.

From this point of view we see why it is that experience may be said to teach an expert in a given field, not only what uniformities have been observed in that field, but what approximate and probable right one has to generalize from the observed to still unobserved uniformities in precisely that region of experience. For the process of sampling tends, in the long run, to correct and to improve itself, so as to show to the expert, although generally not to the layman, what ways of sampling are "fair" in their application to a given region of facts. For

for what he otherwise might not have sought. It directs his attention.

But this, after all, is the least of the services which a good hypothesis renders to science. Its higher service is that, when it is indeed a good type of hypothesis for the field in which it is used, it may be made the starting point of a more or less extended *Deductive Theory*, which enables the investigator to discover indirect means of testing the hypothesis, in cases where direct means fail. One often meets with the remark that a scientific hypothesis must be such as to be more or less completely capable of verification or of refutation by experience. The remark is sound. But equally sound it is to say that a hypothesis which, just as it is made, is, without further deductive reasoning, capable of receiving direct refutation or verification, *is not nearly as valuable to any science as is a hypothesis whose verifications, so far as they occur at all, are only possible indirectly, and through the mediation of a considerable deductive theory*, whereby the consequences of the hypothesis are first worked out, and then submitted to test. If Thales successfully predicted an eclipse, he made and verified a hypothesis. But if this hypothesis was solely founded upon an empirical knowledge of the cycle of former eclipses, his astronomy had not yet passed beyond the statistical stage, and could not pass beyond that stage through even a large number of such verifications. But when a modern astronomer deals with lunar theory, and uses the comparison between theory and observation as, in this case, a very accurate means of testing the degree of accuracy of the Newtonian hypothesis of the law of inverse squares, as the law to which a field of gravitative force is subject, the value of the work done depends upon the vast range of deductive theory which here *separates* the original Newtonian statement from the observed facts. The recorded positions and movements of the moon, when supplemented by the records of the known eclipses that were recorded by the ancients, constitute a very vast "sample" of the physical facts about the moon's motion. The computations which lunar theory makes possible constitute a still vaster "sample" of special results of the Newtonian theory, as applied to the moon. Now if the Newtonian theory of gravitation had only a chance, or a temporary, or a superficial relation to the observable motions of the moon, the chances are extremely small that a very large sample of the

the expert is one who has had experience of many samples of different *ways of sampling* in his own field.

§ 12. Herewith we are prepared to understand a step forward in methodical procedure which took place early in the history of physics, and which has since become possible in very various regions of science. It is obvious that such a step might be expected to consist in some improvement in the choice and in the definition of the regions within which the selection of "fair samples" should be made possible. As Peirce has pointed out, it is just such improvement that takes place when induction assumes the form of *sampling the possible consequences of given hypotheses* concerning the constitution or the laws of some realm of natural phenomena, *or of sampling facts viewed with reference to their relation to such hypotheses.*

The reasoning which is used when hypotheses are tested, is of a fairly well known type. The instance furnished by Newton's hypothesis that a falling body near the earth's surface and the moon in its orbit were alike subject to a force that followed the law of the "inverse squares," has been repeatedly used as an illustration in the text-books of the Logic of Induction. We need not here dwell upon the more familiar aspects of the method of the "working hypothesis" and of its successful verification, or of its correction in the light of observation. Our interest lies in the bearing of the whole matter upon the Theory of Order. This bearing is neither familiar to most minds, nor immediately obvious.

We must therefore sketch the general way in which the union of Theory and Observation is accomplished in the more exact natural sciences, and must then try to show that *what makes this union most effective, depends upon the possibility of defining hypotheses in terms of certain conceptual order-systems whose exactness of structure far transcends, in ideal, the grade of exactness that can ever be given to our physical observations themselves.*

In its simplest form, the method of induction here in question appears as a discovery of natural processes, structures, or laws, through an imaginative anticipation of what they *may be*, and through a testing of the anticipations by subsequent experience. The first and most directly obvious use of an *Hypothesis*, which thus anticipates an observable fact, lies of course in its *heuristic* value. It leads an observer to look results of the theory should agree as nearly as they do with so large a sample of the results of observation. For in such a case *two* samples of facts, the one selected from a realm of *observed physical phenomena*, the other selected from the realm of *the ideal consequences of the Newtonian theory of gravitation*, are compared, not merely in general, but in detail; so that the correspondence of theory with observation is a correspondence of the two samples, so to speak, member by member, each element of each of the two samples approximately agreeing with some element of the other with which, in case Newton's original hypothesis is true, it ought to agree.

§ 13. What here takes place is, *mutatis mutandis*, identical with what constitutes the most important feature in any successful and highly organized combination of Hypothesis, Theory, and Observation. The stages of the process are these.

(1) A Hypothesis is suggested regarding the constitution or the laws of some region of physical fact.

(2) This hypothesis is *such as to permit an extensive and exact Deductive Theory* as to what ought to be present in the region in question, *in case* the hypothesis is true. *The more extensive, exact and systematic the theory thus made possible proves to be, the larger are the possible samples of the "consequences of the hypothesis" which are available*, whenever they are needed, for comparison with the physical facts.

(3) Samples of facts are chosen from a field open to observation and experiment, and are then compared with the results of theory. The more complete the theory, the larger the range of facts that can be called for to meet the need for comparison.

(4) This comparison no longer is confined (as is the case when the statistical and the comparative methods in their similar forms are used) to noting what proportion, r' per cent., of the members of a sample have a certain relatively simple character q. On the contrary, in case the deductive theory in question is highly developed and systematic, the sample of the results of theory which is accessible for comparison is not only complex, but *has a precise order-system of its own* (is, for instance, a system of ideally exact physical quantities) *which must be approximately verifiable in detail in case the original hypothesis is true.* The comparison of theory and fact is therefore here possible with a minuteness of individual detail which, in case of

successful verification, may make it very highly probable that if the system of real physical facts under investigation has any determinate constitution whatever, its constitution very closely agrees with that which the hypothesis under investigation requires.

It thus becomes obvious that the value of the method here in question very greatly *depends upon the exactness, the order, and the systematic character of the concepts in terms of which the hypotheses thus indirectly tested are defined.* If these concepts are thus exact and systematic, they may permit extended and precise deductions, and the result will be that large samples of the exact consequences of a hypothesis, will be such that they can be compared with correspondingly large samples of the facts of observation and experiment. The comparison of two such samples can then be made, not merely in general, but element by element, minutely, with reference to the Order presented and conceived, and in such wise as to make a chance agreement of theory and fact extremely improbable.

The result will be that the truth of the hypothesis that is tested will still be at best only *probable and approximate*, but the probability will tend to become as great as possible, while the approximation will grow closer and closer as the theory reaches more and more exactness and fulness of deductive development, and as it is confirmed by larger and larger ranges of observations.

An almost ideal union of deductive theory with a vast range of observations is found in the modern doctrine of Energy.

§ 14. In view of the foregoing considerations, we can now readily see that this, the most perfect of the scientific methods, namely *the organized union of Theory with Observation requires for its perfection concepts and systems of concepts which permit of precise and extended deductive reasonings*, such as the Newtonian theory of Gravitation and the modern theory of Energy exemplify. It is a commonplace of Methodology that hypotheses which are stated in *quantitatively precise terms*, especially meet, *at present*, this requirement, and lead to physical theories of the desired type. Our account, following Peirce's view of induction, shows *why*, in general, such theories are so important for the study of nature. The "samples of possible consequences" which they furnish are especially adapted to meet the requirements of a minute comparison, element by element, with the samples of observed facts in terms of which the theories in question are to be tested.

Meanwhile our sketch of the general Theory of Order will hereafter show us that quantitative concepts get their importance for deductive theoretical purposes *simply from the fact that the Order-System of the quantities is so precise and controllable a system. Herein, to be sure, the quantities are not alone amongst conceptual objects*, and it will be part of the business of our later sketch to show that *the two concepts, Exact Deductive Theory and Quantitative Theory, are by no means coextensive.* The prominence of quantitative concepts in our present physical theories is nothing that we can regard as absolutely necessary. There may be, in future, physical sciences that will be highly theoretical, and that will not use quantitative concepts as their principal ones. Yet it is certain that they will use *some exact conceptual Order-System.*

But, however this may be, our result so far is the following one:—

A sketch of Methodology has shown, in the case of the Comparative, and the Statistical Methods, and of the Method which unites Observation and Theory, that all these methods use and depend upon the general concept of the *Orderly Array* of objects of thought, with its subordinate concepts of *Series*, of the *Correlation of Series*, and of special *Order-Systems* such as that of the *Quantities.* All these concepts are essential to the understanding of the methods that thought employs in dealing with its objects. And thus a general review of Methodology leads us to the problems of the Science of Order.

SECTION II.

GENERAL SURVEY OF THE TYPES OF ORDER.

§ 15. WHEN the methodical procedure of any more exact physical science has led to success, the result is one which the well known definition that Kirchhoff gave of the science of Mechanics exemplifies. The facts of such a science, namely, are "*described*" with a certain completeness, and in as "simple," that is, in as *orderly* a fashion as possible. The *types of order* used in such a description are at once "forms of thought," as we shall soon see when we enumerate them, and forms of the world of our physical experiences in so far, but *only* in so far as, "approximately" and "probably," our descriptions of the world of the facts of "possible physical experience" in these terms are accurate. The philosophical problem as to *how and why the given facts of physical experience conform as nearly as they do to the forms of our thought*, is a question that can be fairly considered only when the types of order themselves have been discussed precisely *as* forms of thought, that is as "constructions" or "inventions," or "creations," or otherwise stated, as "logical entities," which our processes of thinking can either be said to "construct" or else be said to "find" when we consider, not the physical, but the logical realm itself, studying the order-types without regard to the question whether or no the physical world exemplifies them.

That this mode of procedure, namely the study of the order-types apart from our physical experience, is important for our whole understanding of our logical situation (as beings whose scientific or thoughtful interpretation of nature is in question), is especially shown by the considerations with which our sketch of Methodology has just closed. For it is notable that *all highly developed scientific theories make use of concepts,—*

such for instance as the quantitative concepts,—*whose logical exactness is of a grade that simply defies absolutely precise verification in physical terms.* The Newtonian theory of gravitation, for instance, can never be precisely verified. For the conception of a force varying inversely with the square of the distance, with its use of the concept of a material particle, involves consequences whose precise computation (even if the theory itself did not also involve the well known, still insurmountable, deductive difficulties of the problem of the gravitative behaviour of three or more mutually attracting bodies), would result in the definition of physical quantities that, according to the theory, would have to be expressed, in general, by irrational numbers. But actual physical measurements can never even appear to verify any values but those expressed in rational numbers. Theory, in a word, demands, in such cases, an absolute precision in the definition of certain ideal entities. Measurement, in its empirical sense, never is otherwise than an approximation, and at best, when absolutely compared with the ideal, a rough one.

Why such concepts, which can never be shown to represent with exactness any physical fact, are nevertheless of such value for physical science, our methodological study has now shown us. *Their very unverifiability, as exactly defined concepts about the physical world, is the source of their fecundity as guides to approximate physical verification.* For what the observers verify are the detailed, even if but approximate correspondences between very large samples of empirical data, and samples of the consequences of hypotheses. The exactness of the theoretical concepts enables the consequences of hypotheses to be computed, that is, deductively predetermined, with a wealth and variety which far transcend precise physical verification, but which, for that very reason, constantly call for and anticipate larger and larger samples of facts of experience such as can furnish the relative and approximate verifications. *It is with theoretical science as it is with conduct. The more unattainable the ideals by which it is rationally guided, the more work can be done to bring what we so far possess or control into conformity with the ideal.*

The order-systems, viewed as ideals that our thought at once, in a sense "creates," and, in a sense "finds" as the facts or "entities" of a purely logical (and not of a physical) world, are therefore to be studied with a true understanding, only when one considers them in abstraction from the "probable" and "approximate" exemplifications which they get in the physical world.

§ 16. Yet the logician also, in considering his order-types, is not abstracting from *all* experience. His world too is, in a perfectly genuine sense, empirical. We have intentionally used ambiguous language in speaking of his facts as *either* his "creations" *or* his "data." For if we say that, in one sense, he seems to "create" his order-types (just as Dedekind, for instance, calls the whole numbers "*freie Schöpfungen des menschlichen Geistes*"), his so-called "creation" is, in this case, *an experience of the way in which his own rational will, when he thinks, expresses itself.* His so-called "creation" of his order-types is in fact a finding of the forms that *characterize all orderly activity, just in so far as it is orderly*, and is therefore no capricious creation of his private and personal whim or desire. In his study of the Science of Order, the logician *experiences the fact that these forms are present in his logical world, and constitute it, just because they are, in fact, the forms of all rational activity.* This synthetic union of "creation" and "discovery" is, as we shall see, the central character of the world of the "Pure Forms."

A survey of the forms of order may therefore well begin by viewing them *empirically*, as a set of phenomena presented to the logician by the experience which the theoretical or deductive aspect of science furnishes to any one who considers what human thought has done. The most notable source of such an experience is of course furnished by the realm of the mathematical sciences, whose general business it is *to draw exact deductive conclusions from any set of sufficiently* precise hypotheses. If one considers the work of Mathematics,—analyzing that work as, for instance, the Italian school of Peano and his fellow workers have in recent years been doing,—one finds that the various Mathematical Sciences use certain fundamental concepts and order-systems, and that they depend for their results upon the properties of these concepts and order-systems. Let us next simply report, in an outline sketch, what some of these concepts and systems are.

§ 17. *Relations.* One "concept," one "logical entity," or (to use Mr. Bertrand Russell's term, employed in his *Principles of Mathematics*) one "logical constant," which is of the utmost importance in the whole Theory of Order, is expressed by the term *Relation.* Without this concept we can make no advance in the subject. Yet there is no way of defining this term *relation* without using other terms that, in their turn, must presuppose for their definition a knowledge of what a relation is. In order, then, not endlessly to wait outside the gate of the Science of Order, for some "presuppositionless" concept that can show us the way in, we may well begin with some observations that can help us to grasp what is meant when we speak of a relation. A formal definition "without presuppositions" is impossible, whenever we deal with any terms that are of fundamental significance in philosophy.

Any object, physical or psychical or logical, whereof we can think at all, possesses *characters*, *traits*, *features*, whereby we distinguish it from other objects. Of these characters, some are *qualities*, such as we ordinarily express by *adjectives.* Examples are *hard, sweet, bitter*, etc. These qualities, as we usually conceive of them, often seem to belong to their object without explicit reference to other objects. At all events they may be so viewed. When we think of qualities, as such, we abstract from other things than the possessors of the qualities, and the qualities themselves. But, in contrast with *qualities*, the *relations* in which any object stands are *characters that are viewed as belonging to it when it is considered with explicit reference to, that is, as in ideal or real company with another object, or with several other objects.* To be viewed as a *father* is to be viewed with explicit reference to a child of whom one is father. To be an *equal* is to possess a character that belongs to an object only when it exists along with another object to which it is equal: and so on.

In brief, *a relation is a character that an object possesses as a member of a collection* (a pair, a triad, an n-ad, a club, a family, a nation, etc.), and which (as one may conceive), would *not* belong to that object, were it not such a member. One can extend this definition from any one object to any set of objects by saying that a relation is a character belonging to such a set when the members of the set are either taken together, or are considered along with the members of still other sets.

It is often assumed that relations are essentially *dyadic* in their nature; that is, are characters which belong to a member of a pair *as* such a member, or to the pair itself as a pair. The relation of a *father*, or that of an *equal*, or that of a *pair of equals*, may be viewed as such a dyadic relation. But, as a fact, there are countless relations which are *triadic*, *tetradic*,—*polyadic*, in any possible way. When, for instance, is an object a *gift*? When, and only when there exists the triad: *giver*, *person or other entity whereto* something is *given*, and *object given*. When is an object a legal debt? Only, in general, when *creditor*, *debtor*, *debt*, and *consideration* or other *ground* for which or by virtue of which the debt has been incurred, exist. So that the *debtor*-relation: "a owes b, to c, for d," is in general a *tetradic* relation. Relations involving still more numerous related objects or terms are frequent throughout the exact sciences.

If a relation is dyadic, we can readily express the proposition which asserts this relation by using the symbol (a R b), meaning: "The entity a stands in the relation R to b." Whenever the proposition (a R b) is true, there is always also a relation, often symbolized by $\breve{\mathrm{R}}$, in which b stands to a. This may be called the inverse relation of the relation R. Thus if: "a is father of b," "b is child of a;" and if one hereby means "child of a father" the relation *child of* is, in so far, the inverse of the relation *father of*.

If a relation is polyadic, then such symbols as R ($a\ b\ c\ d\ \ldots$), meaning "a, b, c, d, etc. (taken in a determinate order or way which indicates the place of each in the relational n-ad in question), stand in the (polyadic) relation R." Thus, with due definition of terms R ($a\ b\ c\ d$) may be used to symbolize the assertion: "a owes b to c for (or in consideration of) d;" and so on.

§ 18. *Logical Properties of Relations.* Relations are of such importance as they are for the theory of order, mainly because, in certain cases, they are subject to exact laws which permit of a wide range of deductive inference. To some of these laws attention must be at once directed. They enable us to classify relations according to various *logical properties. Upon such properties of relations all deductive science depends. The doctrine of the Norms of deductive reasoning is simply the doctrine of these relational properties when they are viewed as lawful characteristics of relations which can guide us in making inferences, and thus Logic as the "Normative Science" of deductive inference is merely an incidental part of the Theory of Order.*

Dyadic relations may be classified, first, as *Symmetrical* and

Non-symmetrical relations. A symmetrical dyadic relation is sometimes defined as one that is *identical with its own inverse relation.* Or again, if S is a symmetrical relation, then, whenever the assertion ($a \ S \ b$) is true, the assertion ($b \ S \ a$) is true, whatever objects a and b may be. The relation of *equality*, symbolized by $=$, is a relation of this nature, for if $(a = b)$, then always $(b = a)$.

If a relation is *non*-symmetrical, various possibilities are still open. Thus, if R be a non-symmetrical relation, and if (c R d), the relation R may be such that the assertion ($d \ R \ c$) is *always excluded* by the proposition ($c \ R \ d$), so that both cannot be true at once of whatever (c, d) one may use as the "terms" of the relation, then, in this case, the relation R is *totally* non-symmetrical. Russell proposes to call such relation Asymmetrical. The relation "greater than" is of this type in the world of quantities. But in other cases the relation R may be such that ($c \ R \ d$) does not exclude ($d \ R \ c$) in *every* instance, but only in certain instances. In the case of different relations, the exceptional instances may be for a given R, unique, or may be many, and may be in certain cases determined by precise subordinate laws of their own. Thus it may be the law that (c R d) excludes (d R c), unless some other relational proposition (e R$'f$) is true; while if (e R$'f$) is true, then (c R$'$ d) necessitates (d R c); and so on.

Without reference to the foregoing concept of symmetry, the dyadic relations may be classified afresh, by another and independent principle, which divides them into Transitive and Non-Transitive relations. This new division is based upon considerations which arise when we consider *various pairs* of objects with reference to some one relation R. If, in particular, ($a \ R \ b$) and (b R c), the relation R may be such that ($a \ R \ c$) is, under the supposed conditions *always* true, whatever the objects (a, b, c) may be, then in this case the relation R is *transitive.* If such a law does *not* universally hold, the relation R is non-transitive. The relation, *equal to*, is a transitive relation, according to all the various definitions of equality which are used in the different exact sciences. The so-called "axiom" that "Things equal to the same thing are equal to each other" is, in fact, a somewhat awkward expression of this transitivity, which, by definition, is always assigned, in any exact science to the relation $=$. The expression is awkward, because, by the use of "each other" in the so-called "axiom," the *transitivity* of the relation $=$, is so stated as not to be clearly distinguished from the *symmetry* which also belongs to the same relation. Yet *transitivity and symmetry are mutually independent relational characters*. The relations, "*greater than*," "*superior to*," etc., are, like the relation $=$, transitive, but they are *totally non-symmetrical*. The relations "*opposed to*," and "*contradictory of*" are both of them *symmetrical*, but are also *non-transitive*.

Fewer formulations of this general type have done more to confuse untrained minds than the familiar "axiom :" "Things equal to the same thing are equal to each other," because the form of expression used suggests that the relation, $=$, *possesses* its transitivity *because of* its symmetry. Everybody easily feels the symmetry of the relation $=$. Everyone admits (although usually without knowing whether the matter is one of definition, or is one of some objectively necessary law of reality, true apart from our definitions), that the relation $=$ is transitive. The "axiom" suggests by its mode of expression that this symmetry and this transitivity are at least in this case, necessarily united. The result is a wide-spread impression that the symmetry of a relation always implies some sort of transitivity of this same relation,—an impression which has occasionally appeared in philosophical discussions. But nowhere is a sharp distinction between two characters more needed than when we are to conceive them as, in some special type of cases necessarily united, whether by arbitrary definition or by the nature of things.

If some dyadic relation, say X, is *non-transitive*, then there is at least one instance in which the propositions $(d\ X\ e)$ and $(e\ X\ f)$ are both of them true of some objects (d, e, f), while $(d\ X\ f)$ is false. As in the case of the non-symmetrical relations, so in the case of the non-transitive relations, this *non-transitivity*, like the before mentioned *non-symmetry*, may appear in the form of an universal law, forbidding for a given relation R *all* transitivity; or else in the form of one or more special cases where a given relation does not conform to the law that the principle of transitivity would require. These special cases may be themselves subject to special laws. A relation, T, is *totally non-transitive*, in case the two assertions $(a\ T\ b)$ and $(b\ \mathrm{T}\ c)$ if both at once true, exclude the possibility that (a T c) is true. Thus if "a is father of b" and "b is father of c," it is impossible that "a is father of c" should be true. The relation *father of*, is both totally non-symmetrical and also totally non-transitive. That relation between propositions which is expressed by the verb "contradicts," or by the expression "is contradictory of," is symmetrical, but totally non-transitive. For propositions which contradict the same proposition are mutually equivalent propositions. The relation "*greater than*," as we have seen, is transitive, but totally non-symmetrical. The relation = is both transitive and symmetrical. And thus the mutual independence of transitivity and symmetry, as relational properties, becomes sufficiently obvious.

Still a third, and again an independent classification of dyadic relations appears, when we consider the *number of* objects to which one of two related terms can stand, or does stand, in a given relation R, or in the inverse relation $\breve{R}$. If "a is father of b," it is possible and frequent that there should be several other beings, *c*, *d*, etc., to whom *a* is also father. If "*m* is twin-brother of *n*," then, by the very definition of the relation, there is but *one* being, viz. *n*, to whom *m* can stand in this relation. If "*e* is child of *f*," there are *two* beings, namely the father and the mother, to whom *e* stands in this relation. In a case where the estate of an insolvent debtor is to be settled, and where the debtor is a single person (not a partnership nor yet a corporation), then the transactions to be considered in this one settlement may involve many creditors, but, by hypothesis, only one debtor, so far as this insolvent's estate alone is in question. Here, there are then several beings, (p, q, r, etc.), of each of whom the assertion can be made:—"*p* is creditor of *x*." But so far as this one case of insolvency alone is concerned, all the creditors in question are viewed as a *many* to whom only *one* debtor corresponds, as *the* debtor here in question.

The questions suggested by such cases are obviously capable of very variously multiplex answers, according to the relational systems concerned. Of most importance are the instances where some general law characterizes a given relation *R*, in such wise that such questions as the foregoing cases raise can be answered in universal terms. The principal forms which such laws can take are sufficiently indicated by the three following classes of cases:—

1. The relation *R* may be such that, if (*a* R *b*) is true of some pair of individual objects (a, b), then, in case we consider

one of these objects, b, there are or are possible *other* objects, besides a,—objects m, n, etc.,—of which the assertions (m R b), (n R b), etc., are true; while at the same time, if we fix our attention upon the other member of the pair, a, there are other objects (p, q, r) either actual, or, from the nature of the relation R, possible, such that (a R p), (a R q), etc., are true propositions. Such a relation R is called by Russell and others a "*many-many*" relation. The laws that make it such may be more or less exact, general and important. Thus the relation "*1° of latitude south of*" is such a "many-many" relation, subject to exact general laws.

2. The relation R may be such that, when (a R b) is true of some pair (a, b), the selection of a is uniquely determined by the selection of b while, given a, then, in place of b, any one of some more or less precisely determined set of objects could be placed. Thus if "a is sovereign of b," where the pair (a, b) is a pair of persons, and where the relation *sovereign of* is that of some one wholly independent kingdom (whose king's sovereign rights are untramelled by feudal or federal or imperial relationships to other sovereigns),—then, by law, there is only *one* a whereof the assertion: "a is sovereign of b" is true. But if we first choose a, there will be many beings that could be chosen in place of b, without altering the truth of the assertion. A case of such a relation in the exact sciences is the case "a is centre of the circle b." Here, given the circle b, its centre is uniquely determined. But any one point may be the centre of any one of an infinite number of circles. Such a relation R is called a "one-many" relation. Its inverse $\breve{R}$ would be called a "many-one" relation.

3. A relation R may be such that (whether or no there are many different pairs that exemplify it), in case (a R b) is true of any pair whatever, the selection of a uniquely determines what *one* b it is of which (a R b) is true, while the selection of b uniquely determines what a it is of which (a R b) is true. Such a relation is called a "one-one" relation. Couturat prefers the name "bi-univocal" relation in this case. The "one-one" relations, or, as they are often called "one-one correspondences," are of inestimable value in the order systems of the exact sciences. They make possible extremely important deductive inferences, for example those upon which a great part of the modern "Theory of Assemblages" depends.

The various classifications of dyadic relationships that have now been defined, may be applied, with suitable modifications, to triadic, tetradic, and other polyadic relations. Only, as the sets of related terms are increased, the possible classifications become, in general, more varied and complicated. A few remarks must here suffice to indicate the way in which such classifications of the polyadic relations would be possible.

If the symbol S $(a\ b\ c\ d\ldots)$ means: "The objects a, b, c, d, etc., stand in the symmetrical polyadic relation S," then the objects in question can be mutually substituted one for another, *i.e.* the symbols a, b, c, etc., can be interchanged in the foregoing expression, without altering the relation that is in question, and without affecting the truth of the assertion in question. This is for instance the case if S $(a\ b\ c\ d\ldots)$ means: "a, b, c, d, ... are fellow-members of a certain club," or: "are points on the same straight line," so long as no *other* relation of the "fellow-members" or of the "points" is in question except the one thus asserted. In such cases S $(a\ b\ c\ d\ldots)$, S $(b\ c\ d\ a \ldots)$, etc., are equivalent propositions. Such a relation S is polyadic and symmetrical. The relation R, expressed by the symbol R $(a\ b\ c\ d)$, is non-symmetrical (partially or totally) if in one, in many, or in all cases where this relation is thus asserted there is some interchanging of the terms or of the objects,—some substituting of one for another,—which is not permitted without an alteration of the relation R, or a possible destruction of the truth of the relational proposition first asserted. This is the case if R $(a\ b\ c\ d)$ means: "a owes b to c for, or in consideration of d;" or, in a special case "a owes *ten dollars* to c for *one week's wages*." Such a relation is non-symmetrical. The number of terms used greatly increases the range of possibilities regarding what sorts of non-symmetry are each time in question; since, in some cases, *certain* of the terms of a given polyadic relational assertion can be interchanged, while others cannot be interchanged without an alteration of meaning or the change of a true into a false assertion. Thus if the assertion R $(a\ b\ c\ d)$ means "a and b are points lying on a certain segment of a straight line whose extremities are c and d,"—then a and b can be interchanged, and c and d can be interchanged, without altering the truth or falsity of the assertion; but if the pair (a, b) is substituted for the pair (c, d), and conversely, the assertion would *in general* be changed in its meaning, and might be true in one form, but false when the interchange was made. Consequently we have to say, in general, that a given polyadic relation, R, is symmetrical or non-symmetrical *with reference* to this or that pair or triad or other partial set of its terms, or with reference to this or that *pair of pairs*, or *pair of triads*, of its terms ; and so on. In case of complicated order-systems, such as those of *functions* in various branches of mathematics, or of *sets of points*, *of lines*, etc., in geometry, the resulting complications may be at once extremely exact and definable, and very elaborate, and may permit most notable systems of deductive inferences.

In place of the more elementary concept of transitivity, a more general, but at the same time more plastic concept, in terms of which certain properties of polyadic relations can be defined, is suggested by the process of *elimination*, so familiar in the deductive inferences of the mathematical sciences. Suppose $R\ (a\ b\ c\ d)$ is a tetradic relation, symmetrical or non-symmetrical ; suppose that the relation is such that if the propositions $R\ (a\ b\ c\ d)$ and $R\ (c\ d\ e\ f)$ are at once true, then $R\ (a\ b\ e\ f)$ necessarily follows. A very notable instance of such a relation exists in the case of the "entities of Pure Logic" of which we shall speak later. We could here easily generalize the concept of transitivity so as to say that this relation R is "transitive by pairs." But such transitivity, as well as the transitivity of a dyadic relation, is a special instance of a general relational property which *permits the elimination of certain terms that are common to two or more relational propositions, in such wise that a determinate relational proposition concerning the remaining terms can be asserted to be true in case* the propositions with which we began are true. Let the symbol α represent, not necessarily a single object, but any determinate pair, triad, or n-ad of objects. Let β represent another such determinate set of objects, and γ a third set. Let R and R' be polyadic relations such that $\mathrm{R}\ (\alpha\ \beta)$ and $\mathrm{R}'\ (\beta\ \gamma)$. The first of these symbols means the assertion : "The set of objects consisting of the combination of the sets α and β (taken in some determinate mode or sequence), is a set of objects standing in the relation R." The second symbol, viz. $R'\ (\beta\ \gamma)$ is to be interpreted in an analogous way. Hereupon, suppose that either always, or in some definable set of cases, the propositions $R\ (\alpha\ \beta)$ and $R'\ (\beta\ \gamma)$, if true together, imply that $\mathrm{R}''\ (\alpha\ \gamma)$, where

R'' is some third polyadic relation, which may be, upon occasion, identical with either or both of the foregoing relations, R' and R. In such a case, the information expressed in $R\ (\alpha\ \beta)$ and $R'\ (\beta\ \gamma)$ is *such as to permit the elimination of the set or collection* β, so that a determinate relational proposition, results from this elimination. It is plain that transitivity, as above defined, is a special instance where such an elimination is possible.[1]

With regard to the "one-one," "many-one" and "many-many" classification of dyadic relations, we may here finally point out that a vast range for generalizations and variations of the concepts in question is presented, in case of triadic, and, in general, of polyadic relations, by the "operations" of the exact sciences,—operations which have their numerous more or less "approximate" analogues in the realm of ordinary experience. These operations make possible deductive inferences whose range of application is inexhaustible.

An "operation," such as "addition" or "multiplication," is (in the most familiar cases that are used in the exact sciences) founded upon a *triadic* relation. If $R\ (a\ b\ c)$ means "The sum of a and b is c," or in the usual symbolic form, $a + b = c$, then the triadic relation in question is that of two numbers or quantities to a *third* number or quantity called their "sum." As is well known, the choice of two of these elements, namely the choice of the a and b that are to be added together (the "summands"), determines c uniquely, in ordinary addition. That is, to the pair (a, b) the third element of the triad (a, b, c) *uniquely corresponds*, in case $R\ (a\ b\ c)$ is to be true. On the other hand, given c, the "sum," there are in general, various, often infinitely numerous, pairs (d, e), $(f, g,)$, etc., of which the propositions, $d + e = c$, $f + g = c$, etc., may be true. But in case of ordinary addition if c, the "sum," is first given, and if then *one* of the "summands," say a, is given, the other, say b, can always be

[1] In the closing chapter of his *Psychology*, in a beautiful sketch of the psychological aspects of scientific thinking, Professor Wm. James characterizes the transitivity of those dyadic relations, which are so often used in the natural sciences, by saying that the objects whose relations are of this transitive type follow what he calls "The axiom of skipped intermediaries." This is a characteristically concrete way of stating the fact that *one* main deductive use of transitivity, as a relational property, lies in the fact that it permits certain familiar *eliminations*. If, namely: "a is greater than b, and b is greater than c," *we may eliminate the intermediary* b, and conclude deductively that a is greater than c. We are here concerned, in our text, with the fact that dyadic transitivity is only a special instance of the conditions that make elimination in general possible, and that determine a whole class of Norms of deductive inference.

found (if the use of "negative" numbers or quantities is indeed permitted in the system with which we are dealing), and, when found, is then uniquely determined. Triadic relations, such as that which characterizes addition, may therefore be subject to precise laws whereby, to one element, or to two elements which are to enter into a triad, either one or many ways of completing the triad may correspond, these possible ways varying with the relational proposition whose truth is to be, in a given case, asserted or denied, or is to remain unchanged through the substitution of various new objects for those already present in a given triad.

The "operations" of the exact sciences are of inestimable importance for all the order-systems in terms of which precise theories are defined and facts are described. It is not necessary that they should precisely resemble, in their relational properties, either the "multiplication" or the "addition" of the ordinary numbers and quantities. A glance at their possible varieties (as these are discussed in connection with modern "group-theory," or as a part of the treatment of the various "algebras" which newer mathematics has frequent occasion to develop), will readily show to any thoughtful observer the absurdity of the popular opinion, still often entertained by certain philosophical students, that "mathematics is the science of quantity." The "quantities" are objects that are indeed vastly important. Their "order-system" is definable in terms of a few important properties of certain dyadic and triadic relations. *All our power to reason deductively about quantity depends upon these few relational properties*, whose consequences are nevertheless inexhaustibly wealthy. But the algebra of quantity is *one* only of infinitely numerous algebras whose operations are definable in terms of triadic relations. And there is no reason why other operations should not be defined in terms of tetradic, and, in fact of n-adic relations. The "Algebra of Pure Logic" is, in fact, as Mr. Kempe has shown, the symbolism of a system whose "operations" are superficially viewed, triadic, but are really founded upon tetradic relations (see § 24, below). And mathematical science includes within its scope the deductive reasonings possible in case of all these order-systems, and capable of being symbolized by all these algebras.

§ 19. *Classes.* In describing relations and their properties, we have inevitably presupposed the familiar concept of a *set* or *collection*, *i.e.* of a *class* of objects as already known. *Relations are impossible unless there are also classes.* Yet if we attempt to define this latter concept, we can do so only by presupposing the conception of *Relation* as one already understood. As we have already pointed out, such a "circle in definition" is inevitable in dealing with all philosophical concepts of a fundamental nature.

The concept of a Class or Set or Collection or Assemblage (*Menge*) of objects, is at once one of the most elementary and one of the most complex and difficult of human constructions. The apparent commonplaces of the Socratic-Platonic Methodology, and their intimate relation to the profound problems of the Platonic Metaphysic, which we touched upon in § 3, have shown us from the outset how the most obvious and the deepest considerations are united in this problem. The "burning questions" of the new "Theory of Assemblages" as they appear in the latest logical-mathematical investigations of our days, illustrate surprisingly novel aspects of the same ancient topic.

The concept of a Class, in the logical sense, depends (1) Upon the concept of an *Object*, or *Element* or *Individual*, which *does or does not belong to a given class*; (2) Upon the concept of the *relation* of *belonging to*, i.e. *being a member of* a class, or of *not so belonging*; (3) Upon the concept of *assertions*, true or false, which *declare that* an object is or is not a *member of* a given class; (4) Upon the concept of a Principle, Norm, or Universal which enables us to decide which of these assertions are true and which are false.

The *first* of these concepts is in many ways the most problematic of all the concepts used in the exact sciences. What constitutes an Individual, what is the "principle of individuation," how are individuals known to exist at all, how are they related to universal types, how they can be identified in our investigations, or how they can be distinguished from one another, whether they can be "numerically distinct" and yet wholly or partially similar or identical,—these are central problems of philosophy, which we in vain endeavour to escape by asserting in the usual way that "individuals are presented to us as empirical objects, by our senses." Whoever has had occasion to study any problem involving the doubtful or disputed *identity* of any individual object, knows that *nc* direct sense-experience ever merely presents to us an individual object such as we conceive of, where we subject our processes of identification to exact rules and tests.

For *logical* purposes, an Individual Object is one that we *propose to regard at once as recognizable* or *identifiable throughout some process of investigation, and as unique within the range of that investigation, so that no other instance of any mere kind of object suggested by experience, can take the precise place of any one individual*, when we view ourselves as having found any individual object. Thus *to propose to treat* an object as always *recognizable* under certain conditions, and as such that no *substitute* for it is possible, in so far as we treat it as *this* individual,—all this involves an *attitude of will* which our sense-experience can illustrate and more or less sustain, but *can never prove to be necessary*, or present to us as successfully and finally warranted by mere data.

The concept of an individual is thus one whose origin and meaning are due to our will, to our interest, to so-called pragmatic motives. We actively *postulate* individuals and individuality. We do not merely find them. Yet this does not mean that the motives which guide our will in this postulate are wholly arbitrary, or are of *merely* relative value. *There are some active and voluntary attitudes towards our experience which we cannot refuse to take without depriving ourselves of the power to conceive any order whatever as present in our world.* Without objects conceived as unique individuals, we can have *no Classes.* Without classes we can, as we have seen, define *no Relations*, without relations we can have *no Order. But to be reasonable is to conceive of order-systems, real or ideal. Therefore, we have an absolute logical need to conceive of individual objects as the elements of our ideal order systems.* This postulate is the condition of defining clearly any theoretical conception whatever. The further metaphysical aspects of the concept of an individual we may here ignore. *To conceive of individual objects is a necessary presupposition of all orderly activity.*

An individual once postulated as present may be *classed with* other individuals. If the various individuals in question are *viewed as if they were* already given, the act of *classing them thus*, that is of asserting that these individuals belong in the same class, is again an act of will. Its value is so far *pragmatic.* We accomplish in this way some *purpose* of our own, some purpose of treating things as for some special reason distinguished or, on the other hand, undistinguished. In this sense, *all classes are subjectively distinguished from other classes by the voluntarily selected Norms, or principles of classification which we use. Apart from some classifying will, our world contains no classes. Yet without classifications we can carry on no process of rational activity, can define no orderly realm whatever, real or ideal. In this sense, the act of defining at least some norms or principles of classification is an act whose logical value is not only pragmatic, but also absolute.* For a world that we might conceive as wholly *without* classes, would be simply no world at all. We could do nothing with it or in it. For to act, consciously and voluntarily, in any way whatever is to classify individuals into the objects that *do* and into those that *do not* concern, meet, serve, correspond to, stimulate or result from each sort of activity. Thus classes are in one sense "creations," in another sense absolute presuppositions of all our voluntary activity, and so of all our theories.

If we have in mind some norm of principle of classification, this norm inevitably defines at least one pair of classes, namely a given class and its *negative* or *contradictory* class. For if the class x is defined by a given norm, then the same norm defines the class consisting of whatever objects are not x, a class here to be symbolized by $\bar{x}$.

Whenever we set out to classify any region of our world, real or ideal, we of course always do so because we know, or at least postulate, that there are *some* individuals in that region to be classified. And considered with reference to a given norm, which defines a class x, these individuals will belong *either to x or else to $\bar{x}$.* But of course our norm does not of itself tell us whether there are any individuals, in the region to be classified, which *are* of the class x. We can, then, define a norm for a class x, and later discover that "Everything is $\bar{x}$," so that "There are no x's." In general, then, when we define by its norm the class x, either one of two assertions may turn out to be true about x. Either (1) "x has *no* member," or (2) "x has at least one member." Of these two assertions *one* is true, the other false, when uttered about any determinate class x. That is, these assertions are *mutually contradictory.*

A very vast range of the assertions of the exact sciences can be said to be of one or the other of these two comparatively simple types. A class that has *no* members, a "nothing-class," an "empty class," or "zero-class" may be symbolized by 0. It is in that case a class sharply defined by its norm, but known *not* to contain any of the objects that we have chosen to regard or to define as the individuals of the world (real or ideal) with which we are dealing. If a class x has *no* members, its negative, viz. $\bar{x}$, comprises *everything* that belongs to the realm or (in the phrase of the English logician, De Morgan) to the "universe of discourse" with which we are dealing. The class *everything* can be symbolized by 1. Regarding 0 and 1 as classes, and using = as the symbol, in the present case, of the relation of logical *equivalence* or *identity* between any two classes, we can assert, as formally true of any world, which for any reason, we can classify, that :—

$$(1)\ 0 = \bar{1}\,; \quad (2)\ \bar{0} = 1.$$

That is, the class *nothing* and the class *everything* are negatives each of the other, whenever these terms are used of any one "universe of discourse" into which a definite classification has been introduced.

Given any two distinct classes, x and y, defined by different norms or principles of classification, then inevitably, and without regard to whether x and y are, either or both of them "zero," that is "empty" classes, the very definitions of x and of y require that two new *resulting* classes should be present, as classes that may or may not have members, in our classified world. These new classes are: (1) The "Logical Product" of the classes x and y, that is, the class of those objects in our "universe of discourse" that conform *at once* to the norm of x and to the norm of y, and that, therefore, belong at once to both the classes x and y; (2) The Logical Sum of the classes x and y, that is, the class of those objects that conform *either* to the norm of x *or* to the norm of y, and that therefore belong to *one at least* of the two classes (x, y). We symbolize by xy the logical product of x and y, and by $x+y$, their logical sum. In every extended discussion of classes logical sums and products are sure to occur.

Between two classes, p and q, there may or may not exist a certain relation which is of fundamental importance for all study of classes, and so for all exact science. This is the relation of *subsumption*. It is a relation non-symmetrical, but *not* *totally* non-symmetrical. We may symbolize this relation by $-\!\!<$ If $p -\!\!< q$, then whatever conforms to the norm of p conforms to the norm of q; or, as we also may say, the class p is *included in* the class q. If $(p -\!\!< q)$ and $(q -\!\!< p)$ are at once true, then $(p = q)$. In case the relation $(p -\!\!< q)$ holds true, the logical product of p and $\bar{q}$ has *no* members, or in symbols, $p\bar{q} = 0$. The *subsumption* relation is transitive, that is :—

"If $(p -\!\!< q)$ and $(q -\!\!< r)$ then $(p -\!\!< r)$."

As the modern study of the topic has shown, *the entire traditional "theory of the syllogism" can be expressed as a sort of comment upon, and relatively simple application of, this transitivity of the subsumption-relation.* Thus does the theory of the "norms of thought" form merely a subordinate part of the theory of Logical Order.

One relation remains here to be explicitly characterized,—a relation often confounded with the subsumption relation, but carefully distinguished therefrom, in recent times by Frege, Peano, and Russell. It is *the relation in which an individual stands* to the class to which it belongs, and of which it is a member. The school of Peano symbolize this relation by ϵ. Thus, supposing i to be the name of an individual object, the symbol $(i \epsilon x)$ means : "i is a member of, that is, belongs to the class x." Since a class itself can be and sometimes is treated logically as an individual, in case this class is taken *as* one member of a *set of classes* (as, for instance, when one says : "The powers of 2, such as 2^2, 2^3, etc., form a class that is one of the classes of whole numbers"), we can suppose the proposition $x \epsilon y$ to be true of some class x, y being a class of classes. But in such a case, if $(i \epsilon x)$ and $(x \epsilon y)$, then the assertion $(i \epsilon y)$ is, in general, false. So that the ϵ-relation is *non-transitive*, while the relation $-\!\!<$, the subsumption relation is transitive. They are, then, quite different relations.

Any class, x, consists of the individuals, i, i', i'' . . ., whereof the corresponding assertions $(i \epsilon x)$, $(i' \epsilon x)$, etc., are true. From the formal point of view it is thus possible, and in fact, for certain logical purposes necessary, to develop the "Theory of Classes" upon the basis of the "Theory of Propositions." Propositions, themselves, have certain characteristic logical relations, of *contradiction*, *implication*, and so on. To these relations of propositions those relations of classes which we have named, viz. *negation*, *subsumption*, etc., correspond in certain exact ways. There is therefore possible a "calculus of classes;" although the two doctrines have certain notable differences regarding the principles available for deductive purposes in each of them.

The assertions of the type ($i \epsilon x$) upon which classifications may be said to rest, have the aforesaid paradoxical character. They are, namely, the expressions of *postulates*, or voluntary acts, since all classification involves a more or less arbitrary norm or principle of classification. Yet the *laws* to which such propositions, as well as any logical system of classes are subject, are nevertheless exact, are definable (as we have seen) in terms of precise dyadic, triadic and tetradic relations, and *are not in the least arbitrary*. In fact, despite the arbitrariness of each individual classification, the general laws of logic possess an absoluteness which cannot conceivably be surpassed, and lie at the basis of all order-system and of all theory.

The only possible answer to the question as to *how* the absoluteness of the logical principles is thus consistent with the arbitrariness of each of the classifications which we make, lies in saying that the logical principles define precisely the nature of the "Will to act in an orderly fashion" or in other words of the "Will to be rational."

§ 20. *The Types of Order.* The foregoing concepts of Relation, of Relational Properties, and of Classes, have enabled modern mathematicians, and other students of logic, to define in exact terms a surprisingly vast range of order-systems. With almost dramatic suddenness the considerations which may have seemed so varied, disunited and abstract in the foregoing sketch, suddenly give us, when they are once properly combined, an insight into precisely what is most momentous about the order present in the worlds of Number, of Quantity, of Geometry, and of Theoretical Natural Science generally.

For, in the first place, what order-type is universally present wherever there is *any* order in the world? The answer is, *Serial Order.* What is a *Series*? Any row, array, rank, order of precedence, numerical or quantitative set of values, any straight line, any geometrical figure employing straight lines, yes all space, all time,—any such object involves serial order. Serial order may exist in two principal types, the "*open*" series, and the "*closed*" series or *cycle.* Since the latter type of order may be reduced to the former by certain well-known devices, it suffices here to characterize any serial order that is "open," *i.e.* that does not return into itself. So viewed, a *Series* is :—*A class of individuals or elements such that there exists a single relation R, dyadic, transitive, and totally non-symmetrical, while this relation R is of such a nature that, whatever pair (a, b) of distinct elements of the class in question be chosen, either $(a\ R\ b)$ or else $(b\ R\ a)$ is true.* Since the relation R is by definition totally non-symmetrical, $(a\ R\ b)$ and $(b\ R\ a)$ cannot be true at once of any chosen pair of objects belonging to the series defined in terms of R. If we begin with any pair, (c, h), of elements of a given series, the "place" of any other element a or g is determined with reference to c and h by such assertions as $(a\ R\ c)$ and $(c\ R\ g)$, $(g\ R\ h)$, etc., while the transitivity of the relation R *enables us to use such assertions as a basis for deductive inferences whenever two pairs with a common element appear in the course of our determinations.* Chains of inference, eliminations, etc., result. Thus, once more, certain norms of deductive inference are determined by relational properties.

Now in terms of the variations which this definition of *series* permits to be present in the classes and sub-classes of which a series may consist, an infinite variety of distinct serial types can be defined upon the basis of the single definition just stated, and of the logical properties of classes.[1]

The series of the positive whole numbers, for instance, is characterized by the fact that there is *one* member of the class in question, namely the *first*, which stands in a relation R to every other whole number, R being the transitive and totally non-symmetrical relation of "predecessor," while no positive whole number stands in the relation R to this first one; and by the further fact that whatever number (say 2, or n) one chooses, there is *one* number (say 3, or $n+1$) and *only one*, such that, while $(n\ R\ n+1)$ is true, *no* whole number m exists such that $(n\ R\ m)$ while $(m\ R\ (n+1))$. In this case $(n+1)$ is called the *next successor of n*. And thus the relation "next successor"

[1] The use of the foregoing definition, and the classifications of possible serial types which the definition permits, have now become common property. The significance of the definition, and the wealth of ordinal properties that could be stated in terms of it, were gradually brought to light in the latter half of the nineteenth century through the researches of C. S. Peirce, of Dedekind, of Cantor, and of various other logical and mathematical writers. The results have been summed up, and placed in various new lights, in Russell's *Principles of Mathematics*.

is defined in terms of R, and of the absence of intermediates. A further characteristic of the whole numbers is this, that if any property Q belongs to the *first* whole number, and if Q is such that, in case Q belongs to any whole number, n, Q belongs to the "next successor" of n (say to $n + 1$), then Q belongs to *all* of the whole numbers. This characteristic of the whole number series is defined and applied by combining the other relational properties of the series with the logical properties of classes, and *is of the most fundamental importance for deduction throughout mathematical theory.* Thus still another norm of deductive reasoning is established for a certain class of cases.

Such simple considerations concerning classes and relations define then, the series of the whole numbers, and predetermine the inexhaustible wealth of the Theory of Whole Numbers. An extension of such an ordinal series "backwards" gives us the *negative whole numbers.* The series of the "*rational numbers*" can be characterized as to its ordinal type by defining the relation R, for that series, and also by choosing the elements of the series, *so* that whatever pair (i, k) of distinct rational numbers exists, such that $(i\ R\ k)$ is true, there also exists j different from i and from k such that $(i\ R\ j)$ and $(j\ R\ k)$. A series of this type is now called "*dense.*" Upon the basis of the *dense* series of the rational numbers we can define another series, that of the "cuts" or *Schnitte* of the rational numbers. This new series is (in Dedekind's sense) "continuous." It is defined in terms of still another union of a certain sort of classification with the relational properties already in question. This series of the "cuts" of the rational numbers is the series of the "real numbers." And Cantor has worked out a more precise characterization of the properties of the continuous series of "real numbers" (the so-called "arithmetical continuum") by a still further synthesis of the properties of certain sub-classes which such a series contains, with the general properties of the relation R, whereby the series as a whole is determined.

In consequence, mathematical science is now in possession of a complete definition of the "arithmetical continuum" in purely ordinal terms.

But the numbers are not merely subject to *dyadic* ordinal relations. As usually employed in arithmetic and algebra, they are also subject to *triadic* relations, in terms of which the *operations* of ordinary addition $(a + b = c)$, and of multiplication

($pq = r$) are defined. The momentous problem arises as to *how these triadic relations are themselves related to the dyadic ordinal relations of the number-series.* This problem has been attacked with complete success by the modern students of the foundations of mathematics. It has been shown, first, that the simple series of the *whole numbers*, defined as above, is such as to enable us to define *for that series* the operations of the addition and multiplication of its own terms upon the basis of considerations that involve solely the dyadic relational properties of this whole-number series as it stands. That is, in case of a series such as the whole numbers, positive and negative, the triadic relations involved in addition and in multiplication, can be defined in terms of the dyadic relations whereby the series is ordered. But in case of the dense series of the rational numbers, and still more in case of the "arithmetical continuum" of the "real numbers," and again yet more in case of the "complex numbers" of algebra, *such a reduction of the triadic relations of these numbers to the dyadic relations of the whole numbers can be accomplished only indirectly*, by means of special definitions, which enable us to regard these other series and in fact the whole system of the "complex numbers," as derived, through a sort of "logical genesis," from the original whole number series, by a *series of combinations* of the terms, classes, and relations, of the latter series, and by further combinations of the results of these first combinations. All this "genesis" we have not here room to follow. It is enough to say that the result of this research is to show that all the properties which make the numbers of ordinary algebra subject to the endlessly varied operations of *calculation*, can be reduced to properties which depend: (1) Upon the dyadic relations of order which hold in the whole-number system itself, and (2) Upon the properties and ordinal relations of certain derived logical entities (*pairs* of whole numbers, *classes* of these pairs, *pairs* of real numbers, etc.). And in brief, we can say: All the properties of the numbers which are used in ordinary algebra, are properties of their order-system, while this order-system is *indirectly definable* on the basis of the properties of the whole-number system, and of the properties of certain classes and relations of objects which the whole-number system enables us to define.

The number-system of ordinary algebra being once defined, it is possible to deal, in a systematic way, with the problems which are presented by the physical and ideal *Quantities* with which mathematical theories so frequently deal. *Quantities* are objects, either physical or ideal, that fall into series by virtue of relations of the nature of *greater* and *less*. They have therefore their serial order-systems. They also, in general, are subject to relations of *equality*. In case they are *Intensive* Quantities, their order-systems are definable *only* by means of such dyadic relations, that is, by means of relations of *greater-less*, and by means of the symmetrical relation of *equality*. Extensive Quantities are such as, *over and above* these dyadic relations, of *greater*, *less*, *equal*, are subject to *triadic* relations in terms of which the *sum* of any two extensive quantities that belong to the same system of quantities can be defined. In the realm of the quantities, however, there is no *general* mode of "logical genesis" which makes it possible for us to define triadic relations of the type $(a + b = c)$ upon the sole basis of the dyadic relations *greater*, *less*, and *equal*. Herein the quantities differ, as logical objects, from the number-series viewed as pure algebra views the latter. The "logical genesis" of the rational and of the "real" numbers,—a genesis of which we have just made mention, has no precise and general correspondent process in the world of quantities. Therefore, those triadic relations of most sets of extensive quantities upon which their addition depends, are defined, either (1) upon the basis of empirical inductions (as is the case with physical weights, with quantities of energy, etc.), or (2) upon the basis of voluntarily assumed postulates (as is the case with many systems of ideal quantities, such as for instance the extensive quantities of Pure Metrical Geometry as they are usually treated), or (3) upon some union of postulate and of physical experience (as is frequently the case in the applications of geometry, and in such a science as Mechanics).[1]

Given, however, some workable and sufficiently general definition of a triadic relation upon which an addition-operation can be founded, then the number-system can be at once introduced into the theory of any system of quantities. The exactness of a physical theory of such a *set* of quantities depends

[1] In the very notable case of geometrical theory, a special form of reduction of the "metrical" to the "ordinal" properties of space-forms also exists, whereby the bases of metrical geometry can be indirectly reduced to principles that are stateable wholly in projective, that is, in ordinal terms. This case is of vast importance for the logic of geometry, but cannot further be studied here.

upon such an introduction. The order-system of such a realm of extensive quantities becomes correspondent to the order-system either of a *part* of the numbers, or of the *entire* system of the real or of the complex numbers. Thus, what makes deductive inference in the realm of quantity possible depends solely upon the ordinal properties of this realm.

The application of the foregoing principles regarding serial order-types to the theory and description of more complicated order-systems, involves a set of processes to which we have now made frequent reference, namely: The *Correlation of Series. Upon such correlations the whole theory of Mathematical Functions depends*,—a theory which admits of infinitely numerous variations and applications, and which plays its part in every extended and exact theoretical science. The norms of deductive inference which are definable here are numerous and complex, but vastly important.

The *simplest type of correlation* is that which takes place when a relation of "one-one correspondence" can be established between the members of two series, or between the members of definable parts of such series. In other cases, a "one-many" relation can be established, whereby to every member (say p) of a given series S, there corresponds some determinate number, a pair (q, r) or a triad (q, r, s) of elements, chosen from some series S', or else so that q belongs to S', r to S'', etc.; while, given (q, r, etc.), p is uniquely determined. The possibilities thus suggested may be still further varied without any necessary sacrifice of exactness of definition. In very numerous instances, especially where the operations possible in case of numbers and quantities are in question, we may have a correspondence and correlation of series so established that, to each of a set of pairs (p, q), or of triads (p, q, r) etc. (whereof p shall be chosen from one series, q from another or from the same series, and so on), there corresponds some determinate element x, or some set of elements (x, y, z, etc.), while the element x (or the set x, y, etc.) can be defined as elements of some series or order-system that thus *results from* or *that is definable in terms of the "functional relation*," whose laws lie at the basis of the correlation in question. In general, let α, β, and γ be, not now single individuals viewed merely as such, but *pairs*, *triads*, or other sets of objects. Let the elements whereof each of the sets α, β, and γ, consists, be all chosen in a determinate way from certain series of objects already defined (number-series, points on lines, series of lines or of other geometrical figures, physical quantities, etc.). Suppose that some general law exists which one can state in the form: "If $R'(\alpha)$ and $R''(\beta)$ are both of them true, $R'''(\gamma)$ is true." Then such a law establishes a *functional* relation, or a *system of functional relations*, amongst the various series from which the elements of α, of β, and of γ, respectively, are chosen.

For instance, $R'(\alpha)$ may stand for some combination of quantities of different forms of physical energy (coal burned, water-power supplied, etc.). These forms of energy may be combined in the production of certain industrial products. Each of these quantities, in a special case, will then be a member of its own series (weight of coal, amount of water used at a certain "head," etc.). $R''(\beta)$ may be a combination of the costs of these various forms of energy, when the energy is obtained under certain conditions. And then, again, each of these elements of cost will have its place in its own series, determined by a price-list (price of coal per ton, of water per cubic metre, etc.). Hereupon, in ways determined by the mode of production, by the waste or the use of energy, etc., there may correspond to a given combination $R'(\alpha)$ and a combination $R''(\beta)$, a given set of costs of a set of industrial products, expressed by $R'''(\gamma)$. In such a case the costs of the products will appear as in "*functional relations*" to the sources of energy used, and to the costs of each of these sources. Wherever such a correlation of series, or of sets or systems of series appears, the result is an Order determined by the correlations.

As Klein long since pointed out, the various types of Geometrical Science, the different geometries (metrical, projective, etc.), may be classified in terms of the "invariants" (that is, of the unchanging laws of the results of correlation) to which the different geometrical "transformations" are subject. And the geometrical "transformations" (projections, systematic deformations, dualities, inversions, etc.) involve correlations of sets of series such that (with the foregoing definition of the symbols used), $R'(\alpha)$, and $R''(\beta)$, etc., imply, as their combined result $R'''(\gamma)$, in ways which the relational properties of the geometrical world enable the geometer to define. In general a mathematical "transformation" means a definition of one system of relations by means of a correspondence with other systems of relations and of related terms. Its "invariant" is a law or a relational property or construction which is exemplified by each and by all of the correlated systems.

A very highly important condition of the orderly character of the systems within which such "functional relations," and such "transformations" are possible, is the existence of relational properties that admit of *eliminations*, of the type discussed in our general account of relations (§ 18, near the close). *What transitivity is in the definition of a single series, the more general relational properties which permit elimination are, in the definition of the complex geometrical and physical order-systems which admit of definite and lawfully repeated correlations and transformations.*

It remains to say a word as to the significance of the Symmetrical Relations in the constitution of all such order-types. If a = b = c = d, etc., the set of objects between any two of which such a dyadic symmetrical transitive relation as = obtains, may be called a *Level.* On a topographical map, the lines that indicate levels, the "contour-lines," run through points any two of which stand for physical points, on the surface mapped, such that they are at *equal* heights above some "base-level" (usually above sea-level). Isothermal lines, isobars, parallels of latitude, and countless other symbols for levels, are conspicuous features of the diagrams that are used to depict the orderly structure of real or ideal objects. Yet the members of such a level are not ordered by means of their symmetrical transitive *levelling* relations. They are ordered, if at all, in terms of serial relations, or in terms of the foregoing correlations of systems of series. Yet levelling processes and relations are constantly used in the definition of order-systems. The topographical map, or the "weather map" illustrates this fact. And the vast range of usefulness which the Equation has in mathematics is one of the best known features of that science. Why are relations which by themselves do *not* order, so useful in the definition of types of order?

The answer to this question is three-fold :—

1. The symmetrical relations, and especially the symmetrical transitive relations, enable us to classify, and so *form the basis for all the most exactly definable classifications of the Science of Order.*

2. For this very reason, many of the most important series in the theoretical sciences are *Series of Levels.* Such, for instance, are the series of contour-lines, isobars, etc., on a map.

3. And again, for the same reason, many of the most important *laws* of an ordered world are defined in terms of levels. The " invariants " of a system of " transformations " establish, in general, such levels. That is, when two or more systems are correlated through a " transformation," the results of such correlation leave certain relations that belong to each system unchanged by the passage from one system to the other. Thus a level is established. For instance, the law of the Conservation of Energy is a law expressed by asserting that, between any two states A and B of a given " closed system " in the physical world there obtains a certain symmetrical transitive relation, namely the relation expressed by saying that: " The total energy present in the system in the state A is equal in quantity to the total energy present in the system in the state B." In other words, the total energy remains invariant through the transformation. Thus the statement of the " invariant " law of any system of correlations or of transformations always includes some elements that can be expressed by symmetrical transitive relations. All this is the result of the same *inseparable union of the concepts of Class and of Relation*,—a union which we have illustrated from the beginning of our sketch of Order.

It will be noted, as we now look back, that the various norms of deductive inference, in all the various cases here in question, depend upon the relational properties of the order-systems which are under consideration, and so, in the last analysis, upon the properties of single relations. Thus Formal Logic, as a " Normative science," is incidental to the application of the Theory of Order to this or that process of deductive inference.

SECTION III.

THE LOGICAL GENESIS OF THE TYPES OF ORDER.

§ 21. In our first section, the study of methodology showed us the relation of all scientific procedure to the Theory of Order. In our second section we have portrayed, in a largely empirical fashion, the Types of Order which characterize the exact sciences. Two of the concepts absolutely essential to the Theory of Order, we have already treated, indeed, so as to show *why they are necessary.* These are the concepts of Relation and of Class. For not only are these concepts actually used in the definition of every type of order, but as we have seen, *their necessity depends upon the fact that without them no rational activity of any kind is possible.* We have consequently insisted that these concepts unite in a very characteristic way—"creation" and "discovery," an element of contingency and an element of absoluteness. That a particular physical or psychical *relation*, such as that of father and child, should be present in the world, is as empirical a fact as the existence of colors or tones. That there should be physical objects to *classify*, this again is a matter of experience. And furthermore, every classification of real or of ideal objects is determined in any special instance, by a norm or principle of classification which we voluntarily choose. And in so far classifications are arbitrary, and may be said to be "creations" or "constructions." Yet, whatever else the world contains, if it only contains a reasonable being who knows and intends his own acts, then this being is aware of a certain relation, the relation between performing and not performing any act which he considers in advance of action. And thus relations amongst acts are in such wise necessary facts, that whoever acts at all, or whoever, even in ideal, contemplates *possible courses of action*, must regard at least some of these relations as present in the realm of his conceived *modes of action.* In a similar fashion, as we have seen, every sort of action determines a kind of classification of some world, physical or ideal. In so far, therefore, as the nature of relations and of classes in general determines the existence and the meaning of types of orderly activity, these types of orderly activity, and the order-systems which express their nature, are both empirical objects, "found" (since we experience their presence in our world); and are also *necessary* objects, because if we try to conceive that they are *not* there, our very conception involves modes of action, and hence restores these necessary relations and classes to the world from which we had tried to banish them. We "construct" relational systems and classes in our ideal world. But we also "find" that at least *some* of these constructions are necessary.

A frequently asserted modern view, to which we have made some reference in the foregoing, namely the view called Pragmatism, asserts that all truth, including logical truth, has its basis in the fact that our hypotheses, or other assertions, prove to be *successful*, or show by their empirical workings that they meet the needs which they were intended to meet. From this point of view the logical hypothesis: "That there are classes, relations, and order-systems," would be true merely in so far as the acts of conceiving such objects, and of treating them as real, have, under the empirical conditions under which we do our thinking, a successful result. And thus logical truth, and the logical existence and validity of classes, of relations, and of the various types of order, would stand in the same position in which all the "working hypotheses" of an empirical science stand. These order-systems would exist, and their laws would be valid, precisely in so far as such ways of actively conceiving of the world have successful workings.

But, in the foregoing, we have already indicated that, so far as the existence of classes and of relations in general is in question, and in so far as the validity of certain logical laws is concerned, we are obliged to maintain a position which we may characterize by the term Absolute Pragmatism. This position differs from that of the pragmatists now most in vogue. There are *some* truths that are known to us *not* by virtue of the special successes which this or that hypothesis obtains in particular instances, but by virtue of the fact that *there are certain modes of* *activity, certain laws of the rational will, which we reinstate and verify, through the very act of attempting to presuppose that these modes of activity do not exist, or that these laws are not valid.* Thus, whoever says that there are no classes whatever in his world, inevitably classifies. Whoever asserts that for him there are no real relations, and that, in particular the logical relation between affirmation and denial does not exist, so that for him *yes* means the same as *no*,—on the one hand himself asserts and denies, and so makes a difference between *yes* and *no* ; and, on the other hand, asserts the existence of a relational *sameness* even in denying the difference between *yes* and *no*.

In brief, whatever actions are such, whatever types of action are such, whatever results of activity, whatever conceptual constructions are such, that the very act of getting rid of them, or of thinking them away, logically implies their presence, are known to us indeed both empirically and pragmatically (since we note their presence and learn of them through action); but they are also absolute. And any account which succeeds in telling what they are has absolute truth. Such truth is a "construction" or "creation," for activity determines its nature. It is "found," for we observe it when we act.

It consequently follows that whoever attempts to justify the existence of any of the more complicated systems of order that we have been describing in the foregoing section, has a right to seek for some absolute criterion, whereby he may distinguish what systems of order are necessary facts in the world,—that is, in the world that the logician has a right to regard as necessary,—and what, if any, amongst these forms are either capricious, and unnecessary, or else are suggested by the particular facts of experience in such wise as to remain merely contingent.

The logician's world is the world of hypotheses, and of theories, and of the ideal constructions that are used in these theories and hypotheses. Now theories and hypotheses may be merely suggested to us by physical phenomena, so that, if we had different sensations from our present ones, or if our perceptions followed some other routine than the observed one, we should have no need for these hypotheses and resulting theories. In so far, the hypotheses are contingent, and the theories have only conditional value. Furthermore, some of our activities are indeed arbitrary, so that we may, as the common expression is, "do as we like." And when such modes of activity play their part in the choice or in the definition of our hypotheses, the logician cannot regard them as necessary. But such logical facts as the difference between *yes* and *no*, are not dependent on the contingent aspect of our sensations, but on our rational consciousness of what we intend to do or not to do. Such facts have not the contingency of the empirical particulars of sense. And some modes of action, such as affirmation and denial, are absolute modes.

We can indeed suspend the process of affirmation and denial, but only by suppressing every rational consciousness about what we ourselves purpose to do. The particular deed may be arbitrary. But the absolute modes of activity just suggested are not arbitrary. We cannot choose to do without them, without seeking to choose, since choice is action, and involves, for instance, the aforesaid difference between affirming and denying that we mean to do thus and thus.

§ 22. Considerations of this sort show us that the Theory of Order must undertake a task which the foregoing sketch has only suggested. It now appears that the logician's world has in it *some* necessary elements and laws upon which order-systems may be founded. But this fact does not of itself suffice to tell us *what ones* amongst the enormously complicated order-systems of mathematics include contingent and arbitrary elements, and what ones are indeed in such wise necessary that whoever knows what his own orderly activity is, must recognize that these order-systems belong to his logical world. Let us illustrate the issue thus brought to our attention.

In the physical world, we meet with the difference between *greater* weights and *less* weights. We meet with this difference empirically, and test it by experiment. The result is that we get tests, such as the balance, whereby we can arrange physical weights in a series of Levels, each level consisting of observed weights any two of which are equal, while the series of these levels is determined by the transitive and totally non-symmetrical relation of greater and less. The familiar operations of putting two weights in one scale-pan of a balance and finding a single weight that, put into the other scale-pan, will balance them, enables us to define for the weights an operation of *summation*,—a *triadic relation* of weights. This operation empirically conforms to the laws of the addition of quantities. Hereupon, by processes not further to be followed in this discussion, we establish an ideal and hypothetical correlation between physical weights and the number-system of arithmetic; and so the physical world, so far as weights are concerned, is conceived in orderly terms, in a way that makes many physical theories logically possible.

Now it is obvious that the existence of physical weights, and that all of the aforegoing relations, so far as they are physical relations, are, from our human point of view, both empirical and contingent. We can easily conceive of a physical world without any such phenomena. For if all our knowledge of nature came to us through sight and smell alone, in the form of colours, odours, etc., and if we never saw anything that suggested to us the comparing of weights, we should of course know of no physical facts that would define for us this order-system.

On the other hand, in defining the system of the weights as in the case of any other extensive quantities, we use our empirical facts for the sake of establishing some kind of correlation between the quantities of our physical world and the facts and laws of the number-system. *But what shall we say about the number-system itself?* It is a system, whose first principles can be stated as hypotheses of a very general nature concerning objects that can be distinguished, numbered, etc. Is our experience of the existence of such objects altogether as contingent as our experience of the existence of weights in the physical world? One obvious answer is suggested by the fact that we can apply the system of the whole numbers to characterize our own acts. Any orderly succession of deeds in which we pass from one to the next has certain of the characters of the series of ordinal whole numbers. In any orderly activity that we begin, we have a first act followed by a second, followed by a third, and so on. It therefore may occur to our minds that our knowledge of at least the whole numbers, like our knowledge of the difference between *yes* and *no*, may be founded upon the consciousness of our own activity and some of its necessary characters. But this view, when first stated, meets with the very obvious difficulty that, during our actual human lives, we perform only at best a very limited number of distinct acts, while the whole number-series, as the mathematician conceives it is an *infinite* sequence. Furthermore, nothing about the empirical nature of our activity as human beings seems to determine the number of deeds that we shall do in our short lives. But the whole numbers of the mathematician present themselves as an order-system such that every member of the series *must* have its next successor. No mere observation of the contingent sequence of our own empirical deeds can therefore by itself warrant the necessity that the infinite sequence of the whole numbers should have a place in the logician's world at all.

Yet this consideration is, once more, only a suggestion of a difficulty, but not a decisive proof that the whole-number series is devoid of absolute necessity. For perhaps there is indeed something about the nature of our activity, in so far as it is rational,—something which necessitates a *possible* next deed after any deed that has been actually accomplished. And this possibility may prove to have something absolute about it. Such considerations deserve at least a further study.

To sum up:—The order-systems of mathematics are suggested in *some* cases by contingent empirical phenomena. In *other* aspects these order-systems may prove upon analysis to be absolutely necessary facts, in the same sense in which the existence of classes and relations of some sort are necessary facts in our world. And thus may be stated *the central problem of the Theory of Order.* This problem is:—*What are the necessary "logical entities," and what are their necessary laws? What objects must the logician's world contain? What order-systems must he conceive, not as contingent and arbitrary, but as so implied in the nature of our rational activity that the effort to remove them from our world would inevitably imply their reinstatement*, just as the effort to remove relations and classes from the world would involve recognizing both classes and relations as, in some new way, present.

It is precisely in this form that the problem of the theory of order appears to be, at the present time, undergoing a most progressive series of changes, enlargements, and enrichments. The "Deduction of the Categories" is taking on decidedly new forms in recent discussion. The principles that will enable us in the future to make an indubitable endless progress in this field at least possible, remain very briefly to be considered as our sketch closes.

§ 23. Common to all the recent logicians who have dealt seriously with the problem thus defined is the tendency to reduce all the order-systems of mathematics to a form defined, so far as possible, in terms of a *few* simple and necessary "logical entities," and "fundamental hypotheses" about relational properties and about the objects whose relations are in question.

In all the older attempts to characterize the mathematical systems of an orderly type, great stress was laid upon the assumption of so-called "self-evident" *Axioms*. The example of Euclid in his Geometry, and the Aristotelian logical theory regarding the necessity of founding all proof upon "immediate" certainties,—these were the paramount influences in determining this tendency. But the more the logician considers the so-called "self-evident" principles of the older mathematical statements, the more reason does he see to condemn self-evidence as in itself a fitting logical guide. *When we call an assertion self-evident we generally do so because we have not yet sufficiently considered the complexity of the relations involved.* And many propositions have been supposed to be self-evident truths that upon closer acquaintance have turned out to be decidedly inexact in their meaning, or altogether incorrect.

In two cases, in the foregoing discussion, we have had occasion to indicate for logical purposes how inadequate the older assumptions regarding the axioms of mathematics and other sciences have been. The first case was presented to us by the presupposition of induction, to the effect that the realm of the objects of possible experience has in any of its definable collections of fact *a determinate constitution*. In mentioning this presupposition in § 10, we stated that it is not self-evident. In § 19 this presupposition appeared in the form of the postulate: *That there are Individuals.* The substantial identity of the two postulates appears upon due reflection. But, as we remarked (in § 19), the postulate: That there are individuals, is too complex to be self-evident, although, upon the other hand, a study of the conception of an individual led us to the assertion, not very fully discussed in this sketch, that this postulate is indeed *at once pragmatic and absolute.* As we said, in our former passage, the principle in question has metaphysical aspects that cannot here be discussed.

At all events, however, we gain, and we do *not* lose, by regarding the postulate of individuality not as "self-evident" but as the expression of an extremely complex, but at the same time fundamental demand of the rational will,—a demand without which our activity becomes rationally meaningless.

The other case of a so-called "axiom" was mentioned in § 18, where we spoke of the principle: That things equal to the same thing are equal to each other. We gain instead of losing when this principle no longer seems self-evident, because we have come to observe that it involves a *synthesis* of the logically independent characters, transitivity and symmetry,—a synthesis which always needs to be justified, either by experience or by definition, or by demonstration, or finally, if that is possible, by the method which we have already applied in dealing with the concepts of class and of relation.

As a fact, therefore, most modern investigators of the Theory of Order have abandoned the view that the fundamental types of order can be defined in terms of merely "self-evident" axioms. These investigators have therefore come to be divided, largely, into two classes: (1) those who, in company with the Pragmatists, are disposed to admit a maximum of the empirical and the contingent into the theory of order; and (2) those who are disposed, like the present writer, to regard the fundamental principles of logic as sufficient to require the existence of a realm of ideal, *i.e.* of possible objects, which is infinitely rich, which contains systems such as the order-system of the numbers, and which conforms to laws that are in foundation the same as the laws to which one conforms when he distinguishes between *yes* and *no*, and when he defines the logical properties of classes and relations.

The writers of the first class would maintain, for instance, that whether or no such distinctions as that between *yes* and *no* have a necessary validity over and above that which belongs to physical objects, such systems as the *ordinal whole numbers* are simply hypothetical generalizations from experience, are empirically known to be valid so far as our process of counting extends, and are regarded in mathematics as absolute, so to speak, *by courtesy*. The field within which such logical empiricists very naturally find their most persuasive instances, is the field presented by geometrical theories. Geometry is a field in which purely logical considerations, and very highly contingent physical facts and relations, have been, in the past, brought into a most extraordinary union, which only recent research has begun to disentangle. Is Geometry at bottom a physical science? Or is it rather a branch of pure logic, the discussion of an order-system or order-systems that possess a logically ideal necessity? The modern discussion of the Principles of Geometry has indeed greatly emphasized the enormous part that a purely logical Theory of Order plays in the development of geometry. But such a theory depends after all upon assumptions. Some of these assumptions, such as the famous Euclidian postulate regarding parallels, appear to some of the writers in question to have an obviously empirical foundation, as contingent as is the physical law of gravitation, and as much subject to verifications which are only approximate as that law is.

Over against these logical empiricists there are those who, however they analyze such special cases as that of geometry, agree with Mr. Bertrand Russell (in his *Principles of Mathematics*) in viewing the pure Theory of Order as dependent upon certain "logical constants." Such "logical constants" Mr. Russell assumes to be fundamental and inevitable facts of a realistically conceived world of purely logical entities, whose relation to our will or activity Mr. Russell would indeed declare to be factitious and irrelevant. Given the "logical constants," Mr. Russell regards the order-systems as creatures of definition; although, from his point of view, definition also appears to be a process by which one reports the existence, in the logician's realm, of certain *beings*, namely, classes, relation, series, orders, of the degrees of complexity described in our foregoing sketch. The Theory of Order for Mr. Russell is the systematic characterization of these creatures of definition. It asserts that the properties of these systems follow from their definitions. And pure mathematics consists of propositions of the type "p implies q," propositions p and q being defined, in terms of the "logical constants," and *so*, that, whatever entities there be (Mr. Russell's "variables") which are defined in terms of proposition p, are also such that proposition q holds of them. In the main Mr. Russell's procedure carries out with great finish ideas already developed by the school of Peano. Mr. Russell's doctrines serve, then, as examples of logical opinions which are not, in the ordinary sense, empiristic.

But the "burning questions" already mentioned as prominent in recent logical theory have shown how difficult it is to make articulate the theory of Mr. Russell, the somewhat similar position of Frege in Germany, and the methods of the school of Peano, without making more "pressing" than ever the question as to *what* classes, series, order types, and systems are to be regarded as unquestionably existing in the world which the theory of order studies, when it abstracts from physical experience and confines itself to the entities and systems of entities which can be defined solely in terms of the "logical constants." There is no doubt of the great advance made in recent times by the writers of this school in actually working out the deductive consequences of certain postulates, when these are once used for the purpose of defining a system. And every such working out is indeed a discovery of permanent importance for the theory of order. To define, for instance, what are called ideal "space-forms," upon the basis of principles more or less similar to, or more or less general than, the postulates of Euclid, is to reach actual and positive results valid for all future Theory of Order. But as the present state of the Theory of Assemblages shows, serious doubts may rise in any one case as to whether such definitions and postulates do not involve latent contradictions, which render the theories in question inadequate to tell us what order-systems are indeed the necessary ones, and what the range of those entities is whose existence can be validated by considerations as fundamental as those which we have already used in speaking of classes and relations in general.

§ 24. One method of escape from the difficulties thus suggested is a way that, in principle, was pointed out a good many years since by an English logician, Mr. A. B. Kempe. In the year 1886, in the *Philosophical Transactions of the Royal Society*, Mr. A. B. Kempe published a memoir on the "Theory of Mathematical Form," in which, amongst other matters, he discussed the fundamental conceptions both of Symbolic logic and of Geometry. The ideas there indicated were further developed, by Mr. Kempe, in an extended paper "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," in the *Proceedings of the London Mathematical Society* for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. Kempe's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest.

In 1905 the present writer published, in the *Transactions of the American Mathematical Society*, a paper entitled "The Relation of the Principles of Logic to the Foundations of Geometry." This paper attempts (1) to show that the principles which Mr. Kempe developed can be stated in a different, and as the author believes, in a somewhat more precise way; and (2) that the principles in question, namely the principles which are involved in any account of the nature of logical classes and their relations, are capable of a restatement in terms of which we can define an extremely general order-system. This order-system is the one which Mr. Kempe had partially defined, but which the present author's paper attempted to characterize and develop in a somewhat novel way. The thesis of that paper, taken in conjunction with Mr. Kempe's results may be restated thus:—

Both classes and propositions are objects without which the logician cannot stir a step. Their relations and laws have therefore, in the foregoing sense, an absolute validity. But, if we state these relations as laws in a definite way, and if we thereupon define a further principle regarding the existence of certain logical entities which in many respects are similar to classes and propositions,—a principle not heretofore expressly considered by logicians,—we hereupon find ourselves forced to conceive the existence of a system called, in the paper of 1905, "The System Σ." *This system has an order which is determined entirely by the fundamental laws of logic, and by the one additional principle thus mentioned.* The new principle in question is precisely analogous to a principle which is fundamental in geometrical theory. This is the principle that, between any two points on a line, there is an intermediate point, so that the points on a line constitute, for geometrical theory, *at least a dense series.* In its application to the entities of pure logic this principle appears indeed at first sight to be extraneous and arbitrary. For the principle corresponding to the geometrical principle which defines dense series of points, does not apply at all to the logical world of propositions. And, again, it does not apply with *absolute* generality to the objects known as classes. But it *does* apply to a set of objects, to which in the foregoing repeated reference has been made. This set of objects may be defined as, "certain possible modes of action that are open to any rational being who can act at all, and who can also reflect upon his own modes of possible action." Such objects as "the modes of action" have never been regarded heretofore as logical entities in the sense in which classes and propositions have been so regarded. But in fact our modes of action are subject to the same general laws to which propositions and classes are subject. That is:—

(1) To any "mode of action," such as "to sing" or "singing" (expressed in English either by the infinitive or by the present participle of the verb) there corresponds a mode of action, which is the *contradictory* of the first, for example "not to sing" or "not singing." Thus, in this realm, to every x there corresponds *one*, and essentially *only* one, $\bar{x}$.

(2) Any pair of modes of action, such for instance as "singing" and "dancing," have their "logical product," precisely as classes have a product, and their "logical sum," again, precisely as the classes possess a sum. Thus the "mode of action" expressed by the phrase: "To sing and to dance" is the logical product of the "modes of action" "to sing" and "to dance." The mode of action expressed by the phrase, "Either to sing or to dance," is the logical sum of "to sing" and "to dance." These logical operations of addition and multiplication depend upon triadic relations of modes of action, precisely analogous to the triadic relation of classes. So then, to any x and y, in this realm, there correspond $x\,y$ and $x+y$.

(3) Between any two modes of action a certain dyadic, transitive and not totally non-symmetrical relation may either obtain or not obtain. This relation may be expressed by the verb "implies." It has precisely the same relational properties as the relation $-\!\!<$ of one class or proposition to another. Thus the mode of action expressed by the phrase, "To sing *and* to dance," *implies* the mode of action expressed by the phrase "to sing." In other words "Singing *and* dancing," implies "singing."

(4) There is a mode of action which may be symbolized by a o. This mode of action may be expressed in language by the phrase, "to do nothing," or "doing nothing." There is another mode of action which may be symbolized by 1. This is the mode of action expressed in language by the phrase "to do something," that is, to act positively in any way whatever which involves "*not doing nothing*." The modes of action 0 and 1 are contradictories each of the other.

In consequence of these considerations, *the modes of action are a set of entities that in any case conform to the same logical laws to which classes and propositions conform.* The so-called "Algebra of Logic" may be applied to them. A set of modes of action may therefore be viewed as a system within which the principles of logical order must be regarded as applicable.

Now it would indeed be impossible to attempt to define with any exactness "the *totality* of all possible modes of action." Such an attempt would meet with all the difficulties which the Theory of Assemblages has recently met with in its efforts to define certain extremely inclusive classes. Thus, just as "the class of all classes" has been shown by Mr. Russell to involve fairly obvious and elementary contradictions, and just as "the greatest possible cardinal number" in the Cantorian theory of cardinal numbers, and equally "the greatest possible ordinal number" have been shown to involve logical contradictions, so (and unquestionably) the concept of the "totality of all possible modes of action" involves a contradiction. There is in fact no such totality.

On the other hand, it is perfectly possible to define a certain set, or "logical universe" of modes of action such that all the members of this set are "possible modes of action," *in case* there is some rational being who is capable of performing some one single possible act, and is also capable of noting, observing, recording, in some determinate way every mode of action of which he is actually capable, and which is a mode of action whose possibility is *required* (that is, is made logically a necessary entity) by the *single* mode of action in terms of which this system of modes of action is defined. Such a special system of possible modes of action may be determined, in a precise way, by naming *some one* mode of action, which the rational being in question is supposed to be capable of conceiving, and of noting or recording in some reflective way any mode of action once viewed as possible. The result will be that any such system will possess its own logical order-type. And some such system must be recognized as belonging to the realm of genuinely valid possibilities by any one who is himself a rational being. The order-type of this system will therefore possess a genuine validity, a "logical reality," which cannot be questioned without abandoning the very conception of rational activity itself. For the question is not whether there exists any being who actually exemplifies these modes of activity in the same way in which "singing and dancing" are exemplified in our human world. The logical question is whether the special sets of modes of action whose logical existence as a set of possible modes of action is required (in case there is any one rational being who can conceive of any one mode of action), is a genuinely valid system, which as such has logical existence.

Now the logical system of such modes of action illustrates a principle, which, as just admitted, *does not apply to the Calculus of Propositions.* Nor does this principle apply, with complete generality, to the Calculus of Classes. But what we may here call the Calculus of Modes of Action, while it makes use of all the laws of the Algebra of Logic, also permits us to make use of the principle here in question, and in fact, in case a system of modes of action, such as has just been indicated, is to be defined at all, *requires* us to make use of this principle. The principle in question may be dogmatically stated thus: "If there exist two distinct modes of action p and r, such that $p \prec r$, then there always exists a mode of action q such that $p \prec q \prec r$, while p and q are distinct modes of action and q and r are equally distinct." This principle could be otherwise stated thus "for any rational being who is able to reflect upon and to record his own modes of action, if there be given any two modes of action such that one of them implies the other, there always exists at least one determinate mode of action which is implied by the first of these modes of action and which implies the second, and which is yet distinct from both of them." That this principle holds true of the modes of action which are open to any rational being to whom any one mode of action is open, can be shown by considerations for which there is here no space, but which are of the nature heretofore repeatedly defined in this paper. For the question is not whether there actually lives any body who actually does all these things. *That*, from the nature of the case, is impossible. The question is as to the definition of a precisely definable set of modes of action. And this principle holds for the Calculus of possible modes of action, because, as can be shown, the denial of such a principle for a rational being of the type in question, would involve self-contradiction.

Now the consideration developed by Kempe, and further elaborated in the paper of 1905, before cited, may be applied, and in fact must be applied to the order-system of such a determinate realm of modes of action. Such a realm is in fact of the form of the foregoing system Σ. A comparison of Kempe's results with the newer results developed in the author's later paper would hereupon show :—

(1) That the members, elements, or "modes of action" which constitute this logically necessary system Σ exist in sets both finite and infinite in number, and both in "dense" series, in "continuous" series, and in fact in all possible serial types.

(2) That such systems as the whole number series, the series of the rational numbers, the real numbers, etc., consequently enter into the constitution of this system. The arithmetical continuum, for instance, is a part of the system Σ.

(3) That this system also includes in its complexities all the types of order which appear to be required by the at present recognized geometrical theories, projective and metrical.

(4) That the relations amongst the logical entities in question, namely the *modes of action*, of which this system Σ is composed, are not only dyadic, but in many cases polyadic in the most various way. Kempe, in fact, shows with great definiteness that the triadic relations of ordinary logic, which are used in defining "sums" and "products," are really dependent upon tetradic relations into which 0 or 1, one or both may enter. In addition to these tetradic relations the logical order-system also depends for some of its most remarkable properties upon a totally symmetrical tetradic relation that, in the sense described in § 18, is transitive by pairs. These special features of the system of logical entities are here mentioned for the sake of merely hinting how enormously complex this order-system is. The matter here cannot be further discussed in its technical details. The result of these considerations is *that it at present appears to be possible to define, upon the basis of purely logical relations, and upon the basis of the aforegoing principles concerning rational activity, an order-system of entities inclusive not only of objects having the relation of the number system, but also of objects illustrating the geometrical types of order, and thus apparently including all the order-systems upon which, at least at present, the theoretical natural sciences depend for the success of their deductions.*

And so much must here serve as a bare indication of the problems of the Theory of Order, problems which, at the present day, are rapidly undergoing reconsideration and which form an inexhaustible topic for future research. Of the fundamental philosophical importance of such problems no student of the Categories, no one who understands the significance of Kant's great undertaking, no one who takes Truth seriously, ought to be in doubt. The Theory of Order will be a fundamental science in the philosophy of the future.

Editor's Note: In his review of Ruge's *Encyclopedia of the Philosophical Sciences,* Volume I *Logic,* in the *International Journal of Ethics,* Vol. XXIV, 1913-1914, p. 474, Professor C. D. Broad makes the following interesting comment on Royce's article: "The best contribution is undoubtedly Royce's. He alone deals at any length with inductive logic, and his view that induction does not involve the assumption of any laws of nature but only of laws of probability, seems to me sound. The reason that he offers for the advanced state of those natural sciences that can be treated mathematically are also plausible; and it is interesting to note his suggestion that as other kinds of order system beside that of numbers are worked out, we may be able to enjoy the advantages of mathematical methods in regions of investigation where quantitative considerations are impossible. I have less sympathy with his attempt to connect the indefinables of logic and mathematics with possible volitions, and certainly do not think that he makes out his case here. But at this point one of the irritating consequences of this form of literature enters, and he has to refer us to a paper of his for a sketch of his real reasons. Still Royce does good service in referring to Kempe's work on the connection between the fundamental concepts of logic and of geometry, though Kempe's theories are also very fully given in the last volume of Schröder." The reference is to Ernst Schröder's *Algebra und Logik der Relative,* Leipsic: B. C. Teubner, 1895.

Chapter XVII

THE RELATION OF THE PRINCIPLES OF LOGIC TO THE FOUNDATIONS OF GEOMETRY

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Introduction.

In the year 1886, in the Philosophical Transactions of the Royal Society, Mr. A. B. Kempe published *A Memoir on the Theory of Mathematical Form*, in which, amongst other matters, he discussed the fundamental conceptions both of symbolic logic and of geometry. The ideas there indicated were further developed, by Mr. Kempe, in an extended paper *On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points*, in the Proceedings of the London Mathematical Society for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. Kempe's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest. I have been led, therefore, to attempt a restatement of Kempe's logical-geometrical theory. The restatement has led me to conceptions which, although implied in those which Mr. Kempe emphasizes, present a number of aspects which I believe to be novel, so that a considerable part of the present research follows a path of its own. My introductory words will indicate the nature of Kempe's contribution to the problem of the foundations of geometry, the kind of task which his work has set before me, and my own main interest in preparing this paper.

The fundamental ordinal relation of geometry is the relation which can be, at pleasure, described as the triadic relation "between," or as an asymmetrical, transitive dyadic relation, such as "before," or "antecedent to," or "sequent to." Essentially the same relation is at the root of all serial order, and on this basis the logic of such order has lately been elaborately discussed by Mr. Bertrand Russell, in his *Principles of Mathematics*.

The axioms of geometry, as Dr. Veblen has stated them (Transactions of the American Mathematical Society, July, 1904), consist (1) of

* Presented to the Society April 29, 1905. Received for publication May 5, 1905.

assertions characterizing the "between" relation, and duly restricting the application of this relation so far as the "lines" of geometry are concerned, and (2) of existential propositions defining certain entities that shall possess the relation. A similar prominence of asymmetrical transitive relations appears in Dr. HUNTINGTON'S various *Sets of Postulates* for numbers, groups, etc. (Ibid., January and April, 1905).

The algebra of logic may be viewed (as Dr. HUNTINGTON, following Mr. PEIRCE and SCHROEDER, has lately afresh shown in detail), as depending upon the relation of inclusion or subsumption, sometimes symbolized by $\prec$. This relation is dyadic and transitive, and may be either symmetrical or unsymmetrical. Upon the basis of this one relation we can define the various operations of formal logic, such as logical multiplication and addition. If the relation $\prec$ is in a given instance symmetrical, it ensures what is commonly viewed as the "uniqueness" of an entity. That is: $a \prec a$; and if $a \prec b$, while $b \prec a$, than $b = a$ (see Dr. HUNTINGTON'S paper of July, 1904, in these Transactions, for a fuller statement of the various results of these considerations). The relation $a \prec b$, in so far as it obtains between non-equivalent elements, may serve to define linear series: $a \prec b \prec c \prec d$, etc.; where $a \prec c$, and $a \prec d$. In such a series c may obviously be said to lie "between" b and d, and the analogy to the geometrical relation "between" is in so far plain. "Dense," and in fact, continuous linear series of the subsumption type can be conceived after the analogy of point series. But on the other hand, a system of logical classes differs, with respect to linear relations, from a system of points on a line in *two* very notable ways: —

(1) If $a \prec b \prec d$, and if it is also true that $a \prec c \prec d$, any one of the three relations $b \prec c$, $c \prec b$, $c = b$, is indeed *possible;* but, in case of the logical entities, it is also possible that b and c are such that no one of these relations actually holds between these two. Thus, Siberia is included within the Russian Empire, which itself may be viewed as included within the "Eurasian" continent. And Siberia is also included in Asia, which may also be regarded as included within the "Eurasian" continent. These subsumptions are transitive, and in so far linear in their type. Yet the Russian Empire and Asia do not form a pair possessing the relation $\prec$, read in either sense.

(2) If $a \prec b \prec c \prec d$, and if, also, $i \prec b \prec c \prec j$, the relations of i and a, of j and d are similarly left indeterminate. These relations need not be directly expressible in terms of $\prec$ at all. That is, nothing in the logical relations forbids linear series (whether dense, or continuous or not) to have two or more "points," i. e., elements, in common, while any number of the other elements of the series remain entirely distinct. The logical lines, as Mr. KEMPE observes, may intersect any number of times.

For this very reason, however, the system of logical entities may be viewed simply as much more general and inclusive than the system of the points of space. And thus it becomes possible to regard a given space-form as *a selection from amongst the entities present in a system that exemplifies the logical relation* $-\!\!<$. That is: One may view the points of a space as a select set of logical elements, chosen, for instance, from a given "universe of discourse." This thought, whose possible fruitfulness for the logical development of the foundations of geometry I regard as highly notable, is the essential thought at the basis of Mr. Kempe's paper of 1890, cited at the outset of this introduction.

The reason why such a thought seems promising is this: The relations amongst logical entities are, in any case, the most fundamental relations that we know. Experience shows us in the outer world those ordinal space-relations which geometry generalizes in the concept of "between." But our own thinking processes show us the meaning of the logical relation $-\!\!<$. The latter relation, then, is more suited to be the basis for a theory of the logic of an exact science, in case we can only so define and restrict its application that our ideal geometrical relations can come to be viewed as special instances of those forms which we can develop by the use of pure logic.

Mr. Kempe's procedure, in the paper of 1890, is, in bare outline, as follows: He sets out, not by assuming the ordinary algebra of logic, but by defining, through postulates, a purely abstract set of entities called by him the "base-system," and a relation which may be viewed as a generalized "between." The latter is the relation which, in its most general form, is characteristic of what Kempe himself calls, later in his paper, "flat collections" of any number the elements of the "base-system." But the relation first appears as a *triadic* relation, and is so characterized in the postulates. Kempe uses the notation: $ac \cdot b$, to mean the assertion: "a, b, and c, form a 'linear triad,' with b between a and c." So far the expressions used resemble those for Dr. Veblen's generalized relation "in the order abc." But Kempe's linear triad has these fundamental properties: (1) "If $ab \cdot c$, and $a = b$, then $c = a = b$." (2) "If $a=b$, then $ac \cdot b$ and $bc \cdot a$, whatever entity of the system c may be."* In other words, Kempe permits the "between" relation to hold where the related elements are, for all the purposes of the operations of the system, identical; and then he defines the distinctness of elements by means of a restriction of the relations that are permitted to hold in triads of distinct elements. The result is that the "between" relation becomes Dr. Veblen's "in the order," whenever the elements are all distinct.

The other properties of the "between" relation which are in question for

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* I vary a little the order of Mr. Kempe's statement of his principles. The relation $=$ is defined by Mr. Kempe only in a very highly abstract form which I need not here attempt to discuss. Geometrically interpreted, if this relation holds between points, these become identical points.

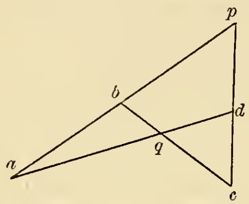
Mr. KEMPE, are obtained by him through assuming two forms of triadic "transversal" propositions as fundamental postulates, viz.: *

I. If two linear triads, $ap \cdot b$ and $cp \cdot d$, exist, such that (as indicated by the notation), b is between a and p, and d is between c and p, then there exists an entity, q, which lies, in a linear triad, between a and d, and, in another linear triad, between b and c.

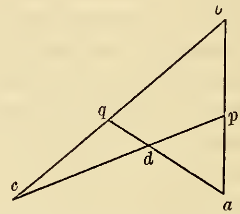
II. If, in the linear triads $ab \cdot p$ and $cp \cdot d$ (as indicated by the notation), p lies between a and b, and d between c and p, then q exists such that q lies, in a linear triad, so that d is between a and q, while, in another linear triad, q is between b and c.

If one interprets these assertions as relating to points in space, they become assertions obviously relating, respectively, to the diagrams following. But, as

I.



II.



they are stated at the outset of Mr. KEMPE's paper, these principles have no specification beyond what the general properties of the linear triad, as just defined, predetermine.

One other existential proposition Mr. KEMPE uses as his *fifth* fundamental principle. This is simply the proposition that any entity belongs to the base-system whose presence there is not inconsistent with the four other principles, —a proposition which of course formally renders the two existential principles, here numbered I and II, superfluous; and which leaves the account of the "base system" inevitably somewhat unsatisfactory.

Mr. KEMPE now proceeds upon this basis, to show, by a decidedly original, although necessarily intricate procedure, that the elements of the base system, as thus defined, possess the properties and relations of a system of logical classes, or of other entities subject to the algebra of logic. In other words, he develops the entire algebra of logic, including the definitions and properties of the operations of logical multiplication and logical addition, without any other assumptions than those simple properties of the "between" relation which have just been stated. The proofs given are such as to apply to any finite number of the elements. Mr. KEMPE leaves, however, some doubt as to infinite collections.

* I vary slightly Mr. Kempe's mode of enunciating these existential propositions at the outset of his paper.

Highly instructive observations are incidental to this development. The system of logical entities appears as possessing a thoroughly symmetrical structure. The "zero"-element and the "universe"-element have no essential distinction from any other similarly related pair of "obverse" elements. Negatives, in general appear as "obverses," because of the symmetrical contrast of their respective relations to the remainder of the system. All the fundamental relations of logic appear as triadic rather than as dyadic. But upon this triadic basis, polyadic relations also develop — the relations of KEMPE's "Flat-collections." These collections, thus named by reason of their resemblance to the various possible configurations of points in an n-dimensional space — "on a line," "on a plane," "in a three dimensional space," etc. — are Mr. KEMPE's means of relating the purely logical to the geometrical entities.

The junction of his principles with the regular algebra of logic once completed, although leaving certain doubts as to the application of his proof to infinite sets, KEMPE proceeds to the geometrical application. By (1) selecting certain "linear sets" of the elements of the base system; and (2) selecting from these sets those which conform to a new principle (here for the first time introduced into the essay), namely, to the principle that any two of the elements of a selected linear set shall determine the whole linear set to be selected, KEMPE is in possession of a system of foundations for a geometry of a "flat space" of n dimensions. The further development of such a geometry is indeed merely sketched in the paper in question. But since the "triangle-transversal" axiom is provided for by the initial principles of the system, and since, by the selection of the linear sets of elements, the ordinary properties of the geometrical "between," and the axiom as to the determination of a line by any two of its points are now also secure, KEMPE's result, although only indicated in his text, is in the main clear. *A space of n dimensions is a select class or set of elements which themselves are entities in a logical "field."* The selection of the entities of a given space is arbitrary; and so the space-forms whose entities are selected may be varied in any way whatever which is consistent with the triangle-transversal-axiom, and with the properties of the generalized between-relation. The problem of the continuity of the geometric sets is only very generally treated, and is not solved.

The wide outlook thus suggested into the theory of space-forms certainly deserves to be better considered than KEMPE's treatment of the subject has so far been considered by mathematicians. For me, however, as a student of philosophy, a still further interest attaches to those results which I have thus suggested, an interest which my mathematical colleagues may also share.

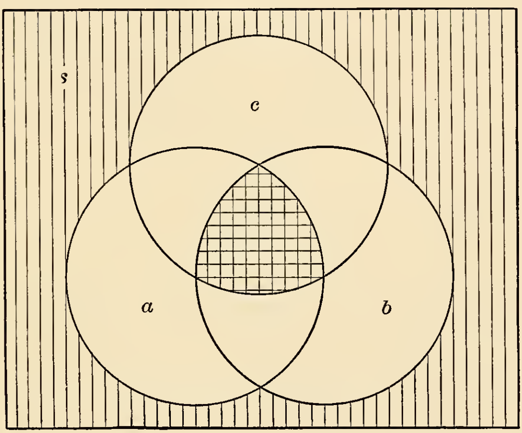
The problem of the foundations of geometry is only a part of that general problem regarding the fundamental concepts of the exact sciences which is now so widely studied. KEMPE's research suggests that, since metrical relations, and therefore (as KEMPE himself briefly indicated in his *Theory of Mathematical Form*, § 309), the whole algebra of ordinary quantity, can be reduced, in any system of three or more dimensions, to a series of propositions based upon purely ordinal relations, — the entire system of the relationships of the exact sciences stands *in a much closer connection with the simple principles of symbolic logic* than has thus far been generally recognized.

Mr. BERTRAND RUSSELL, using very different methods, insists, indeed, in general, upon the closeness of such a connection. But the distinction between the "logic of relations," and the older "logic of classes," and of "propositions," — a distinction which Mr. BERTRAND RUSSELL in his *Principles of Mathematics* regards as something quite fundamental, seems to me to become, in the light of KEMPE's research, a distinction probably quite superficial. Hence to my mind, Mr. KEMPE's theory goes far deeper than Mr. RUSSELL's. Give us a system of entities of the types of logical classes, and we shall find that their relations (all statable in terms of KEMPE's "between"), are already (quite apart from a separate "logic of relations"), certainly as rich as the totality of the relations studied in geometry, and are, for reasons upon which KEMPE has dwelt, probably as rich as the totality of the relations known to the exact sciences, at least so far as the latter have yet been developed. The bare prospect of such a result deserves a careful consideration, in case one takes interest in the unification of the categories of science. KEMPE's theory promises such an unification.

The present memoir proposes to contribute towards a more precise statement of the theory thus outlined. At the basis of my own discussion, I place, however not KEMPE's "between" relation, but another fundamental relation of symbolic logic which has the interest of being absolutely *symmetrical*, while, when it obtains amongst n entities, it permits (upon the basis of certain simple existential propositions), the definition of the properties of KEMPE's "flat collections," and so the definition of asymmetrical relations of a very high degree of complexity. This change of starting point is the prime novelty of the present discussion, as contrasted with KEMPE's.

The *contrast between symmetrical and unsymmetrical relations* seems, to the ordinary view, absolute. Mr. RUSSELL, in his late volume, so treats it. Geometry, and the ordinary algebra of quantity (as these subjects are usually treated), seem to depend on regarding the distinction as quite fundamental. In symbolic logic, however, as Mrs. LADD-FRANKLIN long ago pointed out (in her paper on the algebra of logic in the volume called "*Studies in Logic by members of the Johns Hopkins University*," Boston, 1883), a "symmetrical copula," namely that of "inconsistency," or of "opposition," can be made to accomplish all the work of the ordinary unsymmetrical copula $\prec$. In other words, if I have otherwise defined the meaning of "not," the statement "x is inconsistent with not-y," means the same as "x implies y." The copula in the former case is symmetrical, in the latter unsymmetrical. The former expression makes explicit the "relative product" (as it is called by PEIRCE and RUSSELL) of *two* symmetrical relations (viz., "opposes" and "not"). This "relative product" is, itself, indeed an unsymmetrical relation. But the constituents of this product are symmetrical. This already suggests how asymmetry may be definable in terms of symmetry.

Using as my suggestion some brief observations of KEMPE (in §§ 75–82, of his paper in the Proceedings of the London Mathematical Society), I have therefore chosen to define, by postulates, at the outset of my discussion, a symmetrical relation which I may call "the *O*-relation." This relation is essentially polyadic, and applies at once to any number of terms greater than one. In logical terms, this is the relation in which (if we were talking of the possible chances open to one who had to decide upon a course of action) any set of *exhaustive but, in their entirety, inconsistent choices* would



stand to one another. It is also the relation in which the members of any set of areas stand to one another when there is no area (except the "zero"-area) which is common to *all* the areas of the set at once, while together these areas exhaust some larger surface (which therefore resembles, in its relation to them, a logical "universe"). Thus if, in the annexed diagram, the surface s contains three circles, a, b, and c, and if we then agree to disregard, or to view as stricken out or destroyed, the here shaded portions of the diagram, the circles a, b, and c have then only the stricken-out or "zero" area in common, while together they exhaust what we thus permit to remain of the surface s. In consequence, a, b, and c here form what KEMPE calls "an obverse collection," and what I call, in this paper, an "*O*-collection."

If *two* objects stand in the O-relation each to the other, then these two are related as a pair of "negative" or contradictory classes, or statements, are related to one another. But any number or multitude of objects, in case such are otherwise permitted to exist in a given system, may stand, not in pairs, but as a whole collection, in this relation, and will then constitute what I call an "O-collection." The name that I give to the relation is derived from the close analogy of such collections, even when they contain more than two elements, to KEMPE's pairs of "obverse elements." But I do not myself wish to call the O-collections, with KEMPE "obverse collections," because, as will be seen, I find it convenient to use an expression in my text (where I speak of "mutually obverse" collections), in a way that would make such usage confusing. Hence I read the expression "O-collection," simply as written.

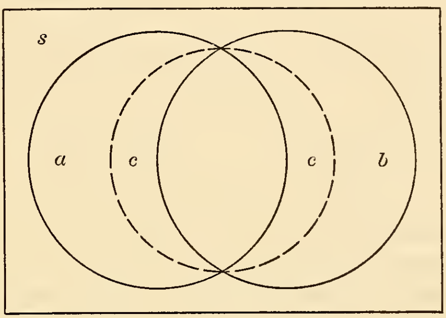
In my text the O-relation is entirely freed from dependence upon all such examples as the ones just used, and is defined solely by postulates, and is to be taken solely as it is there defined. The fact however that it is, in its relational properties, identical with the "yes-no" relation—the earliest exact relation defined by the human mind—is, in this introduction, important. *For what I am in the end to show is that all the serial and other ordinal relations known to logic and to geometry, and all the operations known to both, so far as they are pure exact sciences, are ultimately reducible to assertions that certain entities do, while certain entities do not stand to one another in the perfectly symmetrical O-relation.*

My procedure differs from KEMPE's, not only in making this wholly symmetrical relation, instead of KEMPE's "between," fundamental, but also in the existential principles which I assume. KEMPE's "transversal" axioms form with me a theorem, proved late in the discussion. My own existential principles have to be wide enough to provide for the "continuity" of the system, or, rather, for its inclusion of infinitely numerous continuous systems, and definite enough to make the system of the entities to which the logical calculus is applicable a determinate manifold, inclusive of the points of a space of n-dimensions. The usual treatment of the algebra of logic provides only for arbitrarily determined sets of 2^n or of 2^{2^n} entities in a given logical system. KEMPE calls any such selected set a "full set." KEMPE's further postulate, however, calling for "all entities" consistent with the formal laws, is itself indefinite. In seeking adequate postulates I have been led to two observations which, although in themselves fairly obvious, seem to me to be new, viz., (1) a relation is here shown between the existence of logical sums and products and the general theory of limits and of continuity; and (2) a general definition of the pairs of elements which I have called "conjugate resultants"*

*This concept of "conjugate resultants" is generalized from KEMPE's own generalization of his "unsymmetrical resultants" in § 28 of the essay of 1890. My use of the concept differs in many ways from his.

is made centrally important. The algebra of logic, so far as I know, has not hitherto been brought into definite relations with the problem of the continuum. This is one of the things that I here accomplish. This undertaking involves proving all the principles of logic so as to make them applicable to infinite sets of entities at once. This also I have here done.

Kempe's "linear triad" of elements is represented, in any logical system of classes, by the classes, or areas a, b, and c, which stand in the relation which is represented in the adjoining diagram by the closed figures so lettered. Any area c which includes the common part of a and b, and which is included within their logical sum, is, in Kempe's phrase, such that "c is between a and b." I hereafter symbolize this relation, in my own way, as $F(c|ab)$. The relation in question is called by me the F-relation, because it is that characteristic of



Kempe's "flat collections." The F-relation, so long as "obverses" or "negatives" exist, follows immediately from, and is equivalent to, an O-relation. For, in the diagram if s is the total surface in which a, b, and c are included, then when "c is between a and b," "a, b, and $\bar{c}$ ($\bar{c}$ being the obverse of c) constitute an O-collection," or "are in the O-relation."

The outcome of our discussion will show that, while logical relations can be indifferently stated as O-relations, or stated as F-relations, or (when once addition, multiplication, and negation have been defined) can be stated in terms of equivalence, the F-relations are the only natural means of expressing the geometrical ordinal relations. This difference, however, between the logical and the geometrical entities, is due to the simple fact that (as Kempe points out), when geometrical sets are considered, the obverses of the elements of any set are excluded from that set, so that the obverses may be viewed as ideal elements of the geometry in question. In fundamental meaning all these relations spring from a common root.

If "m is between b and c," I sometimes call m "mediator" or again, on occasion, "resultant" of b and c. I extend the term "resultant" to include the case where a single element stands in an F-relation to any collection of elements.

The axioms or principles assumed at the outset of my discussion, in § 19, are verifiable for a collection of areas all of which are included within a given area, if the O-relation is interpreted as, for the sake of illustration, I have just done. The consistency of these axioms is thus secured from the start. For the verification of "Principle VI" of my set of principles, see § 118 of my text. For KEMPE's term "base-system," I substitute "the system Σ."

The considerable length of the discussion may be justified by the importance, (1) Of a development of the principles of logic solely in terms of a symmetrical polyadic relation; and (2) Of the need of supplementing KEMPE's results by a theory of the continuity of the "base-system."

THE SYSTEM Σ, AND THE O–COLLECTIONS.

CHAPTER I. DEFINITIONS AND PRINCIPLES.

1. The system Σ, whose structure we are to consider in what follows, consists of certain "elements," which we shall regard, in the present discussion, as simple and homogeneous. As symbols for these elements, we shall employ the small letters of the alphabet: $a, b, \cdots, i, j, \cdots, x, y$, etc. In many cases, for the sake of distinguishing one element of a set from others, we shall need subscript marks; and for these too we shall nearly always employ small letters, or if that be convenient, numbers, thus: $a_k, b_i, \cdots, x_1, x_2, p_v$, etc. It is to be noted however, that, unless the contrary is especially indicated, these subscripts are *merely* convenient distinguishing marks; so that the numbers when used as subscripts will, in general, *not possess any ordinal meaning*, but will be used merely as tags. The few exceptions to this rule will explain themselves.

2. The elements thus symbolized may be viewed either singly, or in their collections. A collection of elements will usually be in question, in what follows, as a sort of complex or secondary unit. We shall apply predicates to collections when they are viewed as such complex units, shall compare collections, combine them into larger collections, make partitions of them into the partial collections of which larger collections are composed, classify collections, etc. A collection may consist of a single element of Σ, and is then called a monad, or a monad-collection. A collection of two elements is called a pair, of three a triad, of four a tetrad, of n elements an n-ad. But a collection may consist of an infinite multitude of elements. And, in fact, whenever our statements, and whenever the conditions imposed in the course of a given investigation, do not set definite limits to the multitude of the elements that belong to such collections as are at any time in question, it is always to be understood that the collections

of which we then speak are actually permitted to consist not only of any finite number, but of any multitude whatever of elements.

3. Collections in which a single element, such as b, or x, is viewed as occurring repeatedly, are to be regarded as permissible objects for our consideration; and if we define the number or the multitude of elements in such a collection, or if, for any other reason, we have to treat all the various elements of a given collection in various ways, then, for the purposes of the count, or of the other individual treatment of the elements of any collection, the various repetitions of a given element are to be treated as, in so far, distinct members of the collection in question. Empty, or "zero" collections will not be considered in the present discussion.

4. In order to symbolize a collection without indicating, by the mere symbol, any assertion except the assertion that the collection consists of certain elements, we shall write the symbols for the elements in question, separated by commas, and in a parenthesis. Thus the symbol (a, b) denotes "the pair which consists of the elements a and b." The symbol (x, x) denotes "the pair consisting of x and of x repeated." The symbol (x, x, y, y, r), denotes "the collection consisting of x and of x repeated, and of y and of y repeated, and of r." The symbol $(x_1, x_2, \cdots, x_r, \cdots)$ denotes "the collection consisting of x_1, of x_2, and of an unspecified multitude of other elements, each of which is symbolized by x written with some subscript." In such a case, if no restriction of the multitude of the elements is stated, this multitude of the elements of the collection need not be limited to that of the whole numbers; and the use of whole numbers as subscripts is then of no special significance, beyond that of the convenience of such subscript-symbols. Other subscript-symbols would be equally possible, and may, upon occasion, be used.

5. In many cases, we shall need to symbolize a collection without at the moment designating any of the single elements of which it consists. In such case we shall use Greek letters, α, β, γ, δ, etc. (and, in a few cases, the capital Greek letters also), as symbols for *entire collections*. Thus, the symbol α means "the collection designated as α, consisting of elements which are not hereby further specified." In such a case, the collection α may be, in fact, a perfectly determinate collection, and the symbol α will then be merely a convenient abbreviation. In other cases, α may stand for an unspecified instance of some class of collections; the members of the collection at any time in question being left, by the conditions of the discussion, to be otherwise determined. A collection α or β or δ may be unrestricted as to the multitude of its elements, or may be a monad, a pair, a triad, etc., according as the conditions of a given statement permit or determine.

6. A frequent operation, in our discussion, will be the adjunction of elements to an already given collection, or to elements already under consideration. The

symbol (α, x) will mean "the collection consisting of the collection α of elements together with the element x, adjoined thereto." The symbol (α, β, γ) will mean "the collection formed by adjoining to the collection α all the elements of the collections β and γ." Greek letters, as symbols for whole collections, and small letters a, x, m, etc., as symbols for elements, may thus be combined in the same expression in order to indicate what adjunctions are at any time in question.

7. When we speak simply of "a collection" of the elements of Σ, without further specification of the character of this collection, *the order in which the elements of the collection are named, or otherwise indicated, or in which they stand in the collection, is wholly indifferent.* A collection, in such a general case, *is determined wholly by the fact that certain elements do, while certain elements do not belong to it.* The arrangement of the elements within the collection will concern us only in case the definition of a given type of collections, or the conditions of a given problem, expressly require us to take note of such arrangement.

8. Fundamental, in our discussion of the properties of the system Σ, is a classification of the collections of elements into those which are, and those which are not what we shall call "O-collections." The O-collections form a class of collections whose fundamental properties we define by the laws hereafter stated. The symbol $O(xyz\cdots)$ is to be read as the statement: "The elements x, y, z, etc., taken together, constitute an O-collection." The symbol $O(\alpha)$ is to be read as the statement: "The collection α is an O-collection." The symbol $O(\beta\gamma)$ is to be read as the assertion that "The total collection formed out of the collections β and γ is an O-collection." At pleasure we shall also use the abbreviated form of expression: "the collection $O(xyz\cdots)$," meaning "the collection such that the assertion $O(xyz\cdots)$ is true." The symbol $O(\alpha x)$ means that "the collection formed by adjoining the element x to the collection α is an O-collection." The symbols $O(\alpha\beta\gamma\delta)$, $O(xy\alpha pq\beta jr\delta)$, etc., are to be read so as to assert that whatever total collection of elements and collections is indicated by the letters enclosed in the parentheses, is an O-collection.

9. If a collection is *not* an O-collection, the fact may have to be separately asserted. We propose in the cases where, for the sake of conciseness such usage is advisable, to call the class of all those collections which are *not* O-collections, E-collections. The symbol $E(\alpha)$ may be read at pleasure as the assertion: "The collection α is *not* an O-collection," or again "is an E-collection." Correspondingly we read the symbols: $E(xy)$ (where we also assert that "x and y form an E-pair," or "do not form an O-pair"); $E(x\alpha)$; $E(xy\beta\gamma)$, etc. If α is not an E-collection, then $O(\alpha)$.

In general, we shall speak of "O-pairs," "O-triads," "O-tetrads," "O-n-ads," and of "E-pairs," "E-triads," etc., wherever our collections, whether E-collections or O-collections, are restricted as to the number of elements in a way to which we wish to call attention.

10. When we simply assert $O(\alpha)$ or $E(\beta)$ of any collection, *the order in which the elements of an O-collection or of an E-collection are named or considered is indifferent.* The elements in question in such cases, simply *do* or *do not* belong to the collections in question without regard to the order in which the elements stand.

Equivalent elements.

11. In case two elements are such that *each of them can be substituted for the other in every O-collection in which that other occurs*, while leaving the collection in question still an O-collection, then these two elements are said to be *equivalent each to the other* or to be *mutually equivalent elements.* But if there exists an O-collection into which either of these elements enters, while the other cannot be substituted for the first, in that collection, without altering it into an E-collection, then the two elements are not equivalent. The equation $x = y$ means, therefore, in the present discussion *not* that x and y are identical, but simply that "either x or y may be substituted for the other in any O-collection wherein that other occurs, while the substitution leaves the collection in question still an O-collection." In case it were possible that *neither* x nor y formed a member of *any* O-collection, this definition would imply that they were then also equivalent. The usual properties of the relation of equivalence obviously follow from this definition: viz., $x = x$; if $x = y$, then $y = x$; if x, y, and z are such that $x = y$, and $y = z$, then $x = z$. It is plain that, if $x = y$, either x or y can be substituted for the other in any E-collection in which that other occurs. For if $x = y$, it is, by definition, impossible that $E(x\alpha)$ while $O(y\alpha)$; hence, if $E(x\alpha)$, $E(y\alpha)$ follows; and the converse is also obvious from the definition of equivalence.

12. As just pointed out, equivalent elements need not be identical. Hence, although the assertion: "a is not equivalent to b," obviously implies the assertion: "a is not the same element as b," — these two assertions must still be carefully distinguished, since the second of them does not imply the first. For the assertiou "a is not equivalent to b," we shall use the symbol $a \neq b$. This means that "there exists at least one O-collection into which one of these elements enters, while, if the other is substituted for the first in that collection, the collection in question becomes an E-collection."

Mutually obverse elements and collections.

13. If two elements of Σ, say p and q, are such that $O(pq)$, then p and q are said to be *mutually obverse elements*, or *obverses*, each of the other. Mutually obverse elements are then such elements as together form an O-pair. If the assertion $O(xx)$ were true of any element, that element would be an obverse of itself. A given element may possess various obverses.

14. If a set of O-pairs exists, such that, for certain existent elements of Σ, $O(ab)$, $O(cd)$, $O(ef)$, $O(pq)$, etc. (this set possessing either finite or infinite multitude), and if we suppose a collection α made up by the selection of one member, and of one member only, from each of these pairs, and if a collection β is supposed to be made up out of all the remaining members of the pairs, the collections α and β are said to be mutually *obverse collections.*

15. In consequence of the formation of the collections α and β just defined, certain repetitions of elements may occur in α, or in β, or in both. In such a case, just as in the cases mentioned above, in 3, the various repetitions of any given element are to be regarded as distinct members of the collections α and β in question. Thus, if the O-pair $O(ab)$ is repeated, then, out of the two pairs $O(ab)$ and $O(ab)$, treated, for the purpose in hand, as distinct pairs, we can form, according to the procedure defined in 14, the collections (a, a), and (b, b). These collections are hereupon to be regarded, by virtue of our definition, as mutually obverse collections. Again, if the pairs $O(pq)$, $O(xy)$, $O(mn)$, $O(xk)$, and $O(mq)$, are given, then the collections (p, x, m, x, m), and (q, y, n, k, q), are mutually obverse collections, as are also any two collections that can be formed out of these same pairs by any permissible variation of the procedure defined in 14. The order in which the members of each of the mutually obverse collections are written, is again indifferent.

16. Suppose a collection δ is first given. Let each element of this collection be such that it can be made to form an element of a pair of elements which (whether the elements of this pair are, or are not, repetitions of those present in other pairs), is distinct from the pair of which any other element of δ is a member. If each of the pairs thus formed is an O-pair, then the collection ϵ, consisting of all the remaining elements of the pairs in question, is an obverse of the collection δ, while δ is an obverse of ϵ. Thus, if δ is the collection (x, m, k, l, l), and if $O(xy)$, $O(mj)$, $O(kn)$, $O(lr)$ and $O(ls)$; and if ϵ is the collection (y, j, n, r, s), then the collections δ and ϵ are mutually obverse collections.

Complements and resultants.

17. In case an element q exists such that, for a given collection β, $O(\beta q)$ is true, the element q is called a *complement* of β. In case q and r exist such that, for a given collection β, $O(\beta q)$ is true, while, at the same time $O(qr)$, then r is called a *resultant* of β.

18. The properties of those collections which may be formed of the elements of Σ are, in the main, properties determined by the existence of equivalent and of non-equivalent elements, of obverses and of resultants, together with the existence of certain laws and principles which hold valid for the system.

19. To the statement of these principles or "postulates" we now proceed. They may be classified under two heads. They are: (1) General laws to which

all O-collections, in case such exist, are to conform; and (2) Principles requiring, either conditionally or unconditionally, the existence of certain elements, and of certain collections.

(1) *Laws to which all O-collections conform.*

I. If $O(\alpha)$, then $O(\alpha\gamma)$, whatever collection γ may be.

II. If, whatever element b_n of β be considered, $O(\delta b_n)$, and if $O(\beta)$ is also true, then $O(\delta)$.

(2) *Principles requiring the existence of elements of* Σ.

III. There exists at least one element of Σ.

IV. If an element x of Σ exists, then y exists such that $x \neq y$.

V. Whatever pair (p, q) exists, such that $p \neq q$, r also exists such that, while both $O(rp)$ and $O(rq)$ are false, $O(pqr)$ is true.

VI. If w exists such that $O(\vartheta w)$, then v also exists such that $O(\vartheta v)$, and such, too, that, whatever element t_n of ϑ be considered, $O(vwt_n)$.

20. These principles may be restated, with less use of symbols, thus:

I. An O-collection remains an O-collection, whatever elements or collections may be adjoined to it.

II. If a collection β, consisting wholly of elements which are complements of a collection δ, is an O-collection, then δ itself is an O-collection.

III and IV (in combination). The system Σ contains at least one pair of mutually non-equivalent elements.

V. If any pair of mutually non-equivalent elements is given, a third element of Σ exists which forms an O-pair with *neither* of the elements of this pair, but which is such that the three elements in question together constitute an O-triad.

VI. If there exists any complement of a given collection ϑ, then, if w be such a complement, there exists a complement of ϑ, viz. v, such that every element of ϑ is a complement of the pair (v, w).

At the close of the introduction a system Σ which conforms to all the foregoing principles, has been already pointed out.

Chapter II. Elementary Consequences of the Principles.

21. *The elimination of obverses.* If any collection α is such that x and y exist such that $O(\alpha x)$ and $O(\alpha y)$, while $O(xy)$, then $O(\alpha)$. This follows directly from principle II, if the pair (x, y) be viewed as the collection β of that principle.

22. *The correspondence of mutually obverse O-collections.* If any collection π is such that $O(\pi)$, and if a collection ρ is a collection which is an obverse collection of the collection π, then $O(\rho)$. For let p_n be any element of π. Then in ρ there exists (by the definition of mutually obverse collections, as given

in 14, 15, 16), some element r such that $O(p_n r)$. By principle I, we may adjoin to the O-pair, $O(p_n r)$, all the remaining elements of ρ besides r, and the thus enlarged collection will still remain an O-collection; so that $O(p_n \rho)$. As an analogous result holds of every other element, p_r, of π, without exception, it appears that π consists entirely of elements each of which is a complement of the collection ρ. Since, however, $O(\pi)$, by principle II, $O(\rho)$.

23. *The elimination of common elements.* If η and ϑ are mutually obverse collections; and if x exists such that $O(\beta x)$ and $O(\vartheta x)$, while y exists such that $O(xy)$, then $O(\beta\eta)$. For, by 22, from $O(\vartheta x)$ follows $O(\eta y)$, since, by adjoining x to ϑ, and y to η, we form the two mutually obverse collections (x, ϑ) and (η, y). By adjunction, in accordance with principle I, from $O(\beta x)$ follows $O(\beta\eta x)$ and from $O(\eta y)$ follows $O(\beta\eta y)$. Since $O(xy)$, there follows, from $O(\beta\eta x)$ and $O(\beta\eta y)$, by principle II, as explained in 21, $O(\beta\eta)$.

24. *The elimination of partial collections.* If ϵ consists solely of elements which are complements of a collection λ, if δ and γ are mutually obverse collections, and if $O(\delta\epsilon)$, then $O(\gamma\lambda)$. For ϵ by hypothesis consists of elements which are complements of λ. Let e be, then, an element of ϵ. Then $O(e\lambda)$, and hence, by adjunction (principle I), $O(e\gamma\lambda)$. Furthermore, δ consists wholly of elements which are complements of γ. Hence if d is an element of δ, $O(d\gamma)$ and hence, by adjunction (principle I), $O(d\gamma\lambda)$. Any element of ϵ, and also any element of δ, is thus a complement of $(\gamma\lambda)$. Hence the O-collection $O(\delta\epsilon)$ consists entirely of elements which are complements of the collection (γ, λ). Hence the collection (γ, λ) is itself an O-collection, by principle II. If ϵ reduces to a single element, e, then the hypotheses above stated reduce to $O(\delta e)$ and $O(\lambda e)$, while δ and γ are mutually obverse collections, and the result then becomes identical with that of 23. But if δ reduces to a single element d and γ to an obverse element c such that $O(cd)$, then the result is that if ϵ consists solely of elements which are complements of a collection λ, and if d is such that c exists such that $O(cd)$, while $O(d\epsilon)$, then $O(c\lambda)$.

25. The operations of the reduction of collections through the elimination of elements and of partial collections as explained in the foregoing, will be found to be of fundamental significance throughout our procedure in what follows.

26. *The existence of obverse elements.* By virtue of principles III and IV, there exists (x, y) such that $x \mp y$. Since $x \mp y$, it follows from the definition of equivalence (11) that there exists at least one O-collection into which one of these elements, say x, enters; while in that collection (whether y is also a member of the collection in question or not), y cannot be substituted for x without changing the collection into an E-collection. Let α be the collection thus characterized. Then, by hypothesis, $O(\alpha)$.

If α contains all the elements of Σ, occurring either once each, or in any multitude of repetitions, then a collection exists which contains all of the elements of Σ, and which is an O-collection. If α does not contain all of the elements of Σ, nevertheless, by principle I, since $O(\alpha)$, all of the elements of Σ which do not appear in α may be adjoined to α, and the resulting collection, say ϑ, will be an O-collection.

Hence, in any case, there exists a collection which contains every element of Σ (each element occurring in that collection either once only, or else repeatedly), while this collection, say ϑ, is an O-collection.

27. Since ϑ is such that $O(\vartheta)$, every element of Σ is a complement of ϑ, by principle I. Let w be any element of Σ. Since w is a complement of ϑ, it follows by principle VI, that v exists, such that, whatever element x of Σ, or of ϑ, be chosen, $O(xvw)$. Since every element of ϑ is thus a complement of the pair (v, w), while $O(\vartheta)$, it follows by principle II, that $O(vw)$ is true. By adjoining to the pair (v, w) all the elements of Σ which do not belong to this pair, we now have $O(\Sigma)$, an assertion according to which each element of Σ is supposed to appear once, and without repetition, in the O-collection in question. Since $O(vw)$, v and w are mutually obverse elements.

Since any element whatever of Σ may be taken instead of w, while, each time, an element would be found to take the place here occupied by v, we have so far two results:

(1) The system Σ, taken in its entirety, is an O-collection.

(2) Every element of Σ possesses at least one obverse.

28. A fundamental property of all pairs of mutually obverse elements hereupon comes to our notice, and is as follows: Let b be any element. Let q and r be two obverses of b, so that $O(qb)$ and $O(rb)$. Hereupon let γ be any collection such that $O(\gamma q)$; that is, let the collection $O(\gamma q)$ be any O-collection into which q, one of the obverses of b, enters. Then, by adjunction (principle I), we have, since $O(br)$, $O(br\gamma)$; and, since $O(\gamma q)$, $O(qr\gamma)$ (wherein we may of course change, as we here do, at pleasure, the order in which the members of the O-collection are written). The collection (r, γ) is thus such that, if either of the members of the O-pair $O(bq)$ be separately adjoined to it, the resulting enlarged collection is each time an O-collection. Hence, by principle II, $O(r\gamma)$. It thus appears that, whatever the collection γ may be, if $O(q\gamma)$, $O(r\gamma)$ follows. By a precisely analogous reasoning we could show that if γ is such that $O(r\gamma)$, $O(q\gamma)$ follows. Hence the two obverses of b here in question, viz., q and r, are such that either of them may be substituted for the other in any O-collection in which that other occurs, while still leaving that collection an O-collection. Hence by the definition of equivalence $q = r$. As the reasoning thus used applies to any two obverses of the same element b, whatever b is, we have, as a result, the proposition that *any two obverses of the same element are mutually equivalent elements.* That is, again, if q is an obverse of b, and b is an obverse of r, then $q = r$.

29. If $x = y$, while $O(qx)$ and $O(ry)$, then, by the definition of equivalence, $O(rx)$, and from $O(qx)$ and $O(rx)$ follows, by 28, the proposition $q = r$. Hence *all the obverses of equivalent elements are equivalent.*

30. If $u \neq v$, while $O(nu)$ and $O(mv)$, then $m \neq n$. For if $m = n$, then, by 29, $u = v$. Hence *the obverses of non-equivalent elements are themselves non-equivalent.*

31. It thus appears that all the obverses of the same element, or of equivalent elements, are mutually equivalent, and that non-equivalent elements cannot possess mutually equivalent obverses, still less the same obverse. Accordingly, since equivalence here means capacity for mutual substitution in O-collections, we may henceforth let a *single one* of the obverses of a given element represent the whole class of these obverses, for all the purposes involved in the present discussion of O-collections. This uniquely selected representative of all the obverses of any element x, we shall henceforth regard, therefore, as *the* obverse of x, and as equivalent to the obverse of any element equivalent to x. We shall symbolize this single representative of all these obverses by $\bar{x}$, or, in general, by writing a bar above the symbol of the element of which at any time we define the obverse. $\bar{x}$ cannot be equivalent to the obverse of any element which is not equivalent to x. As the unique representative of the obverses of $\bar{x}$ we may hereupon take an element symbolized by $\bar{\bar{x}}$.

32. By definition, and by 31, $x = \bar{\bar{x}}$; and $\bar{\bar{x}}$ will henceforth be so chosen as to be identically the *same* element as x. The operation of obversion (that is, of finding, for any element x, the unique representative, $\bar{x}$, of the class of elements any one of which forms, with x, an O-pair), hereupon becomes an entirely univocal operation. This operation, if once repeated, is so defined as to be an operation which restores to us the original element.

33. When one collection, δ, is an obverse of another collection, ϵ (see 14, 15, 16), each of these collections, by the substitution of the equivalent elements (in case such substitution is required for the purpose), may be made into a collection consisting wholly of the unique representatives of the obverses of the various members of the other collection. An obverse of the collection δ, thus reduced to the form of a collection of the unique representative obverses of the elements of δ, shall henceforth be symbolized, in our discussion, by $\bar{\delta}$. By the symbol $\bar{\delta}$ we mean, therefore, a certain chosen unique representative of all those collections any one of which is an obverse of the collection δ. The collection $\bar{\bar{\delta}}$ may then be so chosen as to be identical with δ.

34. If $a = b$, then $O(a\bar{b})$ and $O(b\bar{a})$. For $O(a\bar{a})$ by definition (13, 31). Hence, since $a = b$, and since, by the definition of equivalence, we can accordingly substitute b for a in $O(a\bar{a})$, we have $O(b\bar{a})$. And since $O(b\bar{b})$, we have, by the substitution of a for b, $O(a\bar{b})$. On the other hand, if either $O(a\bar{b})$, or $O(b\bar{a})$, is known to be true, then, by the definition of obverses, a is

an obverse of $\bar{b}$, or b is an obverse of $\bar{a}$, as the case may be. But in either case, since obverses of the same element or equivalent elements are equivalent, $\bar{a} = \bar{b}$, and $a = b$.

35. A fundamental characteristic of the system Σ is, further, the fact that: *No monad is an O-collection.* For if $O(x)$, then, by adjunction (principle I), $O(xx)$. By the same principle, if $O(x)$, $O(xq)$, whatever element q may be. Hence any monad q is such that whether x, or an obverse of x (namely x itself), be adjoined to q, always $O(xq)$. Hence, by principle II, $O(q)$. Therefore if a *single* element x exists such that $O(x)$, *every* element of Σ, as for instance q, is such that $O(q)$. Hence, by principle I, whatever collection of elements be adjoined to any element q of Σ, the resulting collection is an O-collection. Hence (by 11), since all possible collections are thus O-collections, *all* the elements of Σ are mutually equivalent. But this contradicts principle IV. Hence it is impossible that any monad x should exist such that $O(x)$. Every monad, therefore, is an E-collection.

36. It will be convenient, at this point, to restate the theorems of 21–24 in the notation for obverse elements which has now been adopted:

(1) If $O(\alpha x)$ and $O(\alpha\bar{x})$, then $O(\alpha)$ (see 21).

(2) If $O(\pi)$, then $O(\bar{\pi})$ (see 22).

(3) If $O(\beta x)$ and $O(\vartheta x)$, then $O(\beta\bar{\vartheta})$ (see 23).

(4) If $O(\delta\epsilon)$, while λ exists such that $O(\lambda e_n)$ for every element e_n of ϵ, then $O(\bar{\delta}\lambda)$ (see 24).

(5) If $O(d\epsilon)$, while λ exists such that $O(\lambda e_n)$ for every element e_n of ϵ, then $O(\bar{d}\lambda)$ (see 24).

37. Any repetitions of an element which occur in an O-collection may be stricken out, so that the element in question occurs but a single time; and the resulting collection will still be an O-collection. For suppose α to be a collection consisting wholly of the element a, repeated any multitude of times. And suppose β to be such that $O(\alpha\beta)$. From $O(\alpha\beta)$ follows by 36 (2), $O(\bar{\alpha}\bar{\beta})$. If any element, either of $\bar{\alpha}$, or of $\bar{\beta}$, be adjoined to the collection (a, β) (which consists of a, occurring only once, with β adjoined), it is plain that the resulting collection will be enlarged so as to constitute an O-collection. For $O(a\bar{a}\beta)$; and, if b_n be any element of β, and $\bar{b}_n$ the obverse of this element $O(a\beta\bar{b}_n)$. The collection $(\bar{\alpha}\bar{\beta})$, consisting entirely of complements of (a, β), is thus an O-collection. Hence $O(a\beta)$, by principle II.

38. It further follows that, if β itself is also a collection consisting solely of a repeated, and if $O(\alpha\beta)$ then the collection $O(\alpha\beta)$ which now consists solely of a repeated, can be reduced to $O(aa)$, and hence to $O(a)$. But $O(a)$ is impossible by 35. Hence no collection consisting solely of repetitions of a single element can be an O-collection. Every such collection must be an E-collection.

It still further follows that, *no element is equivalent to its own obverse.* For since $O(x\bar{x})$, it follows, by the definition of equivalence, that, if $x=\bar{x}$, $O(xx)$, Hence by 37, $O(x)$. But this, by 35, is impossible.

39. If a and b are such that the mutually obverse pairs $(\bar{a}, b)$ and $(a, \bar{b})$ are such that $O(\bar{a}bx)$ and $O(a\bar{b}x)$, then by 21 and 22, $O(\bar{a}b)$. Hence, by 34, $a=b$.

40. If any collection δ is such that $O(\delta)$ is false, so that $E(\delta)$ is true, it follows, from principle I, that if ϵ is any collection such that ϵ consists wholly of elements which belong to the collection δ, while ϵ does not include all of these elements, then $O(\epsilon)$ is false. For if $O(\epsilon)$, then, since δ may be formed from ϵ by adjoining to ϵ certain elements, principle I would require that $O(\delta)$ should be true. Whatever elements, therefore, we omit from an E-collection, δ, the remaining elements form an E-collection. Or, in other words, if a collection is an E-collection, all possible partial collections that can be formed by selecting some of its elements, and omitting others, are also E-collections, so that if ϵ is any such partial collection, $O(\epsilon)$ is false.

41. If $E(a\bar{b})$, then $a \neq b$. For if $a=b$, then, by 34, $O(a\bar{b})$, which contradicts the hypothesis $E(a\bar{b})$. And, on the other hand, if a and b are such that $a \neq b$, then $E(a\bar{b})$, for if not, then $O(a\bar{b})$, and, therefore, by 34, $a=b$, which contradicts the hypothesis $a \neq b$. Thus then, if two elements, x and y, form an E pair, the obverse of either of these elements is not equivalent to the other element; i. e., $x \neq \bar{y}$; and $\bar{x} \neq y$. Plainly, furthermore, if $E(x\bar{y})$, then $E(\bar{x}y)$, and conversely, again, if x exists such that either $E(a\bar{b}x)$ or $E(\bar{a}bx)$ is true, then, by 40, either $E(a\bar{b})$, or else $E(\bar{a}b)$ is true. But as we have just seen each of the assertions: $E(a\bar{b})$ and $E(\bar{a}b)$, implies the other, and also implies $a \neq b$. In the same way, if $E(xyz)$, then $E(xy)$, $E(yz)$ and $E(xz)$. Hence $x \neq \bar{y}$, $y \neq \bar{z}$, etc. In general, if $E(\alpha)$, and if x and y are any two of the elements of α, then, by 40, $E(xy)$, and hence $x \neq \bar{y}$. This is also immediately evident from principle I, and from the definition of obverses. For if $x=\bar{y}$, then $O(xy)$, and then any collection into which both x and y enter is, by principle I, an O-collection.

42. If $E(\alpha)$, and if x be any element whatever, then either $E(\alpha x)$ or $E(\alpha\bar{x})$ must be true. For if neither of these assertions is true, then $O(\alpha x)$ and $O(\alpha\bar{x})$; and then, by principle II, $O(\alpha)$.

43. By 35 every monad is an E-collection. Let x and y be any two elements of the system Σ. Each of these elements possesses an obverse. Since $E(x)$, by 42 either $E(xy)$ or $E(x\bar{y})$ is true; and since $E(y)$, either $E(xy)$ or $E(\bar{x}y)$ is true.

Chapter III. The F-Collections.

44. If two collections, β and η, are such that $O(\beta\bar{\eta})$, then the collections β and η stand to each other in a relation which we shall also, at pleasure, express

by the symbol $F(\beta|\eta)$, wherein the symbols β and η appear with a short vertical line between them. This symbol then, in the first place, expresses precisely the *same* facts that are expressed by the symbol $O(\beta\bar{\eta})$. That is the symbol $F(\beta|\eta)$ may at pleasure be read as the assertion: "The collection consisting of β taken together with the collection $\bar{\eta}$ (the collection which is the obverse of η), constitutes in its totality, an O-collection." Since, by 22 and 33, $O(\beta\bar{\eta})$ implies $O(\bar{\beta}\eta)$, and conversely, the symbol $F(\beta|\eta)$ could equally well be read: — "The collection $(\bar{\beta}, \eta)$ is an O-collection." But the symbol $F(\beta|\eta)$ is especially intended to emphasize the fact that, when $O(\bar{\beta}\eta)$, and consequently when $O(\beta\bar{\eta})$, the collections β and η stand to each other in a relation which is *mediated by the existence of their respective obverse collections.* β and η are then collections such that each, if adjoined to the obverse of the other collection, unites with that obverse, to constitute a total collection that is an O-collection. Expressing this fact with a primary reference to β and η, instead of to β and $\bar{\eta}$, or to $\bar{\beta}$ and η, the symbol $F(\beta|\eta)$ may now be read as the assertion: "The collection β forms, *with* the collection η, a *determinate F-collection.*" The special significance of the adjective *determinate* will appear below. The vertical line is intended as a sort of punctuation mark, to indicate the distinction between the two collections in question.

45. If $O(\gamma)$, and if hereupon γ be in *any* way exhaustively divided into two "partitions," that is, into two mutually exclusive collections of elements δ and ϵ, such that $O(\delta, \epsilon)$ is the same collection as γ, it is plain from the foregoing that $F(\delta|\bar{\epsilon})$. So the same O-collection makes possible various different assertions in terms of determinate F-collections. If γ is a collection of unrestricted multitude, the multitude of the possible assertions in terms of F-collections becomes also unrestricted.

46. The rule for transforming our assertions so that what are explicitly defined as O-collections shall appear in the form of explicitly designated and determinate F-collections, is consequently as follows: If the assertion $O(\gamma)$ is given, and if we are to express this as an assertion regarding some determinate F-collection, then we choose at random any partial collection δ of the elements of γ. Let $\bar{\epsilon}$ be the collection which is the obverse of the collection ϵ, where ϵ consists of all the remaining members of γ, not included in δ. Write $F(\delta|\bar{\epsilon})$, or, at pleasure, $F(\bar{\epsilon}|\delta)$. That is, put δ on one side and $\bar{\epsilon}$ on the other side of the vertical. The way in which the partial collections included in the parenthesis are placed, in so far as these two collections are merely considered with respect to their succeeding or preceding the vertical, is then capable of transposition at pleasure. The resulting expression is to be read, as above defined, and as an assertion in terms of a determinate F-collection. Instead of $F(\bar{\epsilon}|\delta)$, we can equally well write $F(\epsilon|\bar{\delta})$, or $F(\bar{\delta}|\epsilon)$.

47. The rule for the inverse operations transforming an assertion regard-

ing a determinate F-collection into an assertion regarding an O-collection is now obvious. If the assertion $F(\alpha|\beta)$ is given, then we first take the collection $\bar{\alpha}$, or, at pleasure $\bar{\beta}$; that is, we take the obverse collection corresponding that collection which stands on one chosen side of the vertical; and then we combine this obverse collection with that collection which stands on the other side of the vertical. Hereupon we write $O(\bar{\alpha}\beta)$ or $O(\beta\bar{\alpha})$, or $O(\bar{\beta}\alpha)$ or $O(\alpha\bar{\beta})$, at pleasure.

48. In case expressions such as $F(\alpha\beta x|\gamma\delta\bar{y}z)$, or similarly complex symbols appear, we read in this way: "The collections α and β, together with the element x, constitute a collection which, taken as a whole, forms an F-collection with a collection consisting of the partial collections γ and δ, and of the elements $\bar{y}$ and z." The expression just set down asserts the same as is asserted by the symbol $O(\alpha\beta x\bar{\gamma}\bar{\delta}y\bar{z})$, or as is asserted by the symbol $O(\bar{\alpha}\bar{\beta}\bar{x}\gamma\delta\bar{y}z)$.

49. If a collection, λ, consists of some finite number, n, of elements, it is one of a set or group of 2^n collections which can be formed from the given collection by first leaving that collection unchanged, and by then transforming, in every possible way, one, two, three, $\cdots$ and finally all of the n elements of λ into their respective obverses. If one of these 2^n collections is an O-collection, e. g., if the original collection λ is an O-collection; then the collection $\bar{\lambda}$, which is one of the set of 2^n collections, is also an O-collection. All of the other collections of the set are hereby required to be determinate F-collections. But in symbolizing these determinate F-collections, the arrangement of the elements is no longer wholly indifferent. One must, in each case, set upon one side of the vertical all of those elements which, in any one of the transformed collections, are obverses of elements of λ; upon the other side one must place all those elements which are identical with elements of λ. One is then to set the rearranged collection within a parenthesis, and is to write F before this parenthesis. In each case one thus asserts the same fact as is asserted by $O(\lambda)$; but does so, each time, with a different stress upon the partial collections whose relations to one another are thus pointed out. By means of the determinate F-collections, one thus analyzes, in a particular way, the various aspects of the meaning of the assertion $O(\lambda)$. Yet each one of the determinate F-collections points back, infallibly, to the same pair of O-collections; and also predetermines the constitution of all the other determinate F-collections of the same set; so that, in thus emphasizing various aspects of the meaning of the assertion $O(\lambda)$, one still never loses the power to return from one aspect, thus emphasized, to any or to all the other aspects of the same assertion. The determinate F-collections thus defined may be grouped in $(2^n - 2)/2$ pairs; since, in general, if $F(\alpha|\beta)$, $F(\bar{\alpha}|\bar{\beta})$ is true. When we enumerate the set of determinate F-collections, it is sufficient to name one of each pair.

50. *Indeterminate F-collections.* It is, however, occasionally convenient to

express simply the assertion that there exists *some* O-collection $O(\bar{\kappa}\lambda)$, such that the collection (κ, λ) is precisely the same collection as a given collection η, that is, such that (κ, λ) stands for a partition of η; while we nevertheless leave it entirely undetermined *what one* of the possible partitions of the collection η it is with regard to which this assertion holds true. In this case we may write simply $F(\eta)$, a symbol which we read as the assertion: "The collection η is an (indeterminate) F-collection." An indeterminate F-collection may prove, when its determination is specified, to be any one of the determinate F-collections which correspond to the possible partitions of η. Thus, if η is the same collection as (κ, λ) and if $F(\kappa|\lambda)$ is true, $F(\eta)$ is true; while if we merely know that $F(\eta)$ is true, we know that some one of the assertions $F(\kappa|\lambda)$, $F(\kappa'|\lambda')$, etc., is true—where $(\kappa, \lambda)(\kappa', \lambda')$, etc., are various possible partitions of the single collection η. In the same way, if $F(\eta)$, some one of the assertions $O(\bar{\kappa}\lambda)$, $O(\bar{\kappa}'\lambda')$, etc., is true.

If, setting out from the assertion $O(\gamma)$, we consider some possible partition of the collection γ, say (δ, ϵ), and then, instead of writing, as above (45), $F(\delta|\bar{\epsilon})$ we write simply $F(\delta\bar{\epsilon})$, we surrender some of the information conveyed in the original assertion $O(\gamma)$, as well as in the assertion $F(\delta|\bar{\epsilon})$. For it now no longer appears what determinate F-collection corresponds to the indeterminate F-collection $F(\delta\bar{\epsilon})$; and the latter assertion tells us only that some one of the possible partitions of the collection $(\delta, \bar{\epsilon})$ is such that, if it is made (e. g., the partition $(\delta_1, \delta_2, \bar{\epsilon}_1, \bar{\epsilon}_2)$, wherein (δ_1, δ_2) is the same collection as δ, and $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ the same collection as $\bar{\epsilon}$)—then $F(\delta_1\epsilon_1|\delta_2\bar{\epsilon}_2)$, so that $O(\delta_1\bar{\epsilon}_1\bar{\delta}_2\epsilon_2)$. It is plain that the indeterminate F-collections occur in pairs. If $F(\alpha)$, then $F(\bar{\alpha})$.

51. An example will serve to distinguish more clearly the kinds of information conveyed by the assertion that a collection is a determinate and by the assertion that this collection is an indeterminate F-collection. Let $F(ab|cd)$ be true. This is equivalent to asserting $O(ab\bar{c}\bar{d})$. Any one of the possible collections which can be formed by transforming (a, b, c, d) through the substitution of the obverse of one or of more of its elements, is then also a determinate F-collection. Thus the assertion $F(ab|cd)$ requires:

For the collection $(a, b, c, \bar{d})$ the assertion $F(ab\bar{d}|c)$;

For the collection $(a, b, \bar{c}, d)$ the assertion $F(ab\bar{c}|d)$;

For the collection $(a, \bar{b}, c, \bar{d})$ the assertion $F(a\bar{d}|c\bar{b})$;

For the collection $(a, \bar{b}, \bar{c}, \bar{d})$ the assertion $F(a\bar{c}\bar{d}|\bar{b})$;

and so on; while each of these assertions implies, and is implied by $O(ab\bar{c}\bar{d})$.

The assertion $F(abcd)$ does not necessarily imply any one of the foregoing determinate assertions. It indicates that, of the $2^n - 1$ or 15 possible collections other than (a, b, c, d) in the set of collections producible from (a, b, c, d) through the substitution of obverses, some one, and consequently (since $O(\alpha)$ implies $O(\bar{\alpha})$), some *pair* of collections, must be O-collections. It would depend upon this pair to determine how the vertical lines which define the determinate F-collections ought to be distributed in each of the various cases which hereupon arise. Some one of these possible distributions express the truth in each case; but the assertion $F(abcd)$ does not tell us which one this is. Thus, if $F(abcd)$, and if we then consider the collection $(a, \bar{b}, c, \bar{d})$, it is possible that:

$F(a|\bar{b}c\bar{d})$, i. e., $O(ab\bar{c}d)$, and $O(\bar{a}\bar{b}c\bar{d})$; or $F(a\bar{b}c|\bar{d})$, i. e., $O(a\bar{b}cd)$, and $O(\bar{a}b\bar{c}\bar{d})$; or $F(a\bar{d}|\bar{b}c)$, i. e., $O(\bar{a}\bar{b}cd)$ and $O(ab\bar{c}\bar{d})$, and so on for all the other cases.

The assertion $F(abcd)$ requires some pair of these alternative O-assertions or some corresponding pair of the F-assertions, to be true, but does not specify what pair in any of the cases in question.

The indeterminate F-collections, like the O-collections, are perfectly symmetrical. In case of a pair (a, b), the alternative pairs of assertions:—

$$\left\{ (1) \begin{array}{c} O(ab) \\ O(\bar{a}\bar{b}) \end{array} \middle| (2) \begin{array}{c} O(a\bar{b}) \\ O(\bar{a}b) \end{array} \right\}$$

are such that if the pair (1) of assertions are both true, the assertions (2) are both of them false. Hence $F(ab)$ can mean only that $O(\bar{a}b)$ and $O(a\bar{b})$ are both of them true; while $F(a\bar{b})$ means that both $O(ab)$ and $O(\bar{a}\bar{b})$ are true. Hence, in case of pairs of elements, the distinction between determinate and indeterminate F-collections vanishes: and the assertion $F(ab)$ is perfectly deter minate.

Elementary properties of the F-collections: operations and transformations.

52. A number of elementary properties of F-collections, and a survey of certain ways in which they may be transformed, may now be readily obtained from the already established properties of the O-collections.

(1) If $F(xy)$ then, since $O(x\bar{y})$, $x = y$. See 34.

(2) If $F(ax|b)$ and $F(bx|a)$, then $a = b$. For $O(a\bar{b}x)$ and $O(\bar{a}bx)$. See 39.

(3) If $F(\eta)$, then $F(\eta\gamma)$ where γ is any collection whatever (Principle I).

(4) If $F(\eta)$, then $F(\bar{\eta})$, as was already observed in connection with the definition of the F-collections. If $F(\beta|\eta)$, then $F(\bar{\beta}|\bar{\eta})$.

(5) If $F(\eta|x)$ and $F(\eta|\bar{x})$, then $O(\eta)$, since $O(\eta\bar{x})$ and $O(\eta x)$. The question may then arise whether $F(\eta)$ is in a given case, also true. It is here first obvious that $F(\eta)$ does not follow from $F(\eta|x)$ and $F(\eta|\bar{x})$. For instance, if $F(ab|x)$ and $F(ab|\bar{x})$, then $O(ab)$. But, if $O(ab)$, $a = \bar{b}$; while if $F(ab)$, $a = b$ (by (1) of the present paragraph). By 38, however, if $O(ab)$, it is impossible that $a = b$. Hence, if $F(ab|x)$ and $F(ab|\bar{x})$,

$F(ab)$ is false. On the other hand, if $O(\alpha)$, then by adjunction, $O(\alpha x\bar{y})$ and $O(\alpha xy)$. Hence, in this case, $F(\alpha x|y)$ and $F(\alpha x|\bar{y})$. But in this case also since $O(\alpha)$, $O(\alpha\bar{x})$. And so $F(\alpha|x)$, i. e. $F(\alpha x)$, is likewise true. If $F(\eta|x)$ and $F(\eta|\bar{x})$, $F(\eta)$ is accordingly possible; but does not follow herefrom.

53. Any determinate F-collection remains such when transformed according to the following rule: *Substitute* for any element, or for any collection, which stands upon one side of the vertical, *the obverse* of that element or collection, *transfer* the obverse in question to the other side of the vertical, being careful to retain, as the result of the transfer, at least one element on each side of the vertical. Thus, if $F(x\beta|y\delta)$, then $F(\beta|\bar{x}y\delta)$, $F(\bar{y}\beta x|\delta)$, $F(x|\bar{\beta}y\delta)$, $F(\beta\bar{y}\bar{\delta}|\bar{x})$, etc., are all of them true. This is obvious, because all these expressions mean the same as $O(\bar{x}\bar{\beta}y\delta)$ or as $O(x\beta\bar{y}\bar{\delta})$.

This is called transformation by transfer. If the elements are transferred by this rule except that *all* the elements are permitted at the end to stand upon *one* side of the vertical, the vertical can then be omitted; but the resulting collection must be regarded as thus transformed into an O-collection.

54. If $F(x\beta|\bar{x}\gamma)$ then $F(\beta|\bar{x}\gamma)$ and $F(x\beta|\gamma)$. If $F(x\beta|\bar{\beta}\gamma)$, then $F(x\beta|\gamma)$ and $F(x|\bar{\beta}\gamma)$. That is, if the obverse of an element or of a collection, which stands on *one* side of the vertical, itself stands on the *other* side of the vertical, then either of the two mutually obverse collections or elements may be stricken out (by 37). For if $F(x\beta|\bar{x}\gamma)$, then $O(x\beta x\bar{\gamma})$, and so $O(\beta x\bar{\gamma})$. Hence $F(\beta x|\gamma)$ and $F(\beta|\bar{x}\gamma)$. This is called a transformation by means of the omission of superfluous obverses; and the procedure obviously applies to collections as well as to elements.

55. If $F(\alpha|\beta)$, and if all the elements of α are mutually equivalent, and all the elements of β are mutually equivalent, then all the elements of the collection are mutually equivalent.

For $O(\alpha\bar{\beta})$. Let a be one of the elements of α and b of β. Then, by 36, $O(\alpha\bar{\beta})$ reduces to $O(a\bar{b})$, whence follows, by 34, $a = b$.

It is now obvious that any repetitions of an element which occur upon one side of the vertical in a determinate F-collection may be stricken out; and also that, by virtue of principle I, any element may be added to that collection which stands upon either side of the vertical, so that, if $F(\alpha|\beta)$, $F(\alpha y|\beta)$ and $F(\alpha|\beta y)$, where y is any element.

Elimination-theorems for F-collections.

56. If two collections β and δ are such that there exists a collection π such that $F(\delta|\pi)$, while, for every member y_r of the collection π, $F(y_r|\beta)$, then $F(\beta|\delta)$.

For, if $F(\delta|\pi)$, then $O(\delta\bar{\pi})$. And if $F(y_r|\beta)$, then, for every member

$\bar{y}_r$ of the collection $\bar{\pi}$ it is true that $O(\bar{y}_r\beta)$. Substituting for the symbol $\bar{\pi}$ the symbol ϵ, and for $\bar{y}_r$ (the representative symbol for any member of $\bar{\pi}$), the symbol x_r (as the representative symbol for any member of the collection now called ϵ), we have δ and β such that there exists a collection ϵ such that $O(\delta\epsilon)$, while, for every member x_r of ϵ, $O(x_r\beta)$. Hence, by 36, and by virtue of the properties of O-collections pointed out in 24, $O(\delta\bar{\beta})$. Hence $F(\delta|\beta)$ or $F(\beta|\delta)$.

57. In case the collection δ reduces to the single element d, the theorem assumes the following form :

If any collection β forms a determinate F-collection with every member of π, separately considered, while the collection π taken as an entirety, is such as to form a determinate F-collection with an element d, then β forms a determinate F-collection with d.

This theorem permits the elimination of π, in case $F(d|\pi)$, and in case the set of determinate F-collections $F(y_r|\beta)$ is given, where y_r is a variable for which every element of π may separately be substituted.

58. If π reduces to a single member y, we have the result of 56 reduced to the form :

If two collections β and δ are such that there exists an element y such that $F(y|\beta)$ and $F(y|\delta)$, then $F(\beta|\delta)$.

This last result furnishes a means for the direct elimination of an element y common to two determinate F-collections, in case y stands alone, on one side of the vertical, in each collection. Here too we deal with a type of *transitivity* whose consequences are of great importance.

59. If $F(x\beta|y\gamma)$, and $F(y\beta|z\gamma)$, then $F(x\beta|z\gamma)$. For, by transfer (53), from $F(x\beta|y\gamma)$ follows $F(x\beta\bar{\gamma}|y)$. And from $F(y\beta|z\gamma)$ follows $F(\bar{\beta}z\gamma|y)$. Hence, by 58, we can eliminate y, and thus we obtain $F(x\beta\bar{\gamma}|\bar{\beta}z\gamma)$. By 54, we may hereupon transform this F-collection by striking out $\bar{\beta}$ from the right side of the vertical (since β occurs on the left side), and $\bar{\gamma}$ from the left side (since γ occurs on the right side). We thus obtain $F(x\beta|z\gamma)$, which was to be proved.

The transformations and the type of elimination here used are typical of the methods which are to be employed in considering and in combining F-collections. These methods correspond to the adjunctions and eliminations already used in case of O-collections.

60. The result of 59 is the principal theorem relating to the transitivity of the relations involved in F-collections. Its importance justifies a proof directly in terms of the properties of O-collections.

If, namely, $F(x\beta|y\gamma)$, then $O(x\beta\bar{y}\bar{\gamma})$. From this follows, by adjunction, $O(x\beta\bar{\gamma}\bar{y}\bar{z})$. If $F(y\beta|z\gamma)$, then $O(y\beta\bar{z}\bar{\gamma})$. From this follows, by adjunction, $O(x\beta\bar{\gamma}y\bar{z})$. From the two O-pentads, thus formed through adjunction, follows, by principle II, $O(x\beta\bar{\gamma}\bar{z})$. Hence $F(x\beta|\gamma z)$.

61. The hypothesis $F(x\beta|y\gamma)$ sets x in a determinate dyadic relation R to y; and this relation is unsymmetrical, since y does not stand in this relation to x. The hypothesis $F(y\beta|z\gamma)$ sets y in this same relation R to z. The conclusion sets x in the relation R to z. The relation in question is therefore transitive. But the unsymmetrical transitive dyadic relation in question has been entirely derived from the wholly symmetrical relations defined by the O-collections.

62. If β, γ, and δ are given collections, and if y and e are given elements, such that $F(y\beta|\delta)$ and $F(e\delta|\beta)$, while $F(e|y\gamma)$, then $F(\gamma\delta|\beta)$.

For by transfer (53), from $F(y\beta|\delta)$ and $F(e|y\gamma)$ follow the two assertions $F(y|\delta\bar{\beta})$ and $F(y|e\bar{\gamma})$. By 58, it follows that $F(\delta\bar{\beta}|e\bar{\gamma})$; whence follows again by transfer $F(\bar{\beta}\gamma\delta|e)$. From $F(e\delta|\beta)$ follows, by transfer, $F(e|\beta\bar{\delta})$. From this and $F(\delta\bar{\beta}\gamma|e)$ follows, by 58, $F(\beta\bar{\delta}|\gamma\delta\bar{\beta})$. By omission of superfluous obverses (54), we obtain $F(\beta\bar{\delta}|\gamma)$ and $F(\beta|\gamma\delta)$.

63. If $F(x|\beta)$ and $F(y|\beta)$, while $F(xy|\delta)$ then $F(\delta|\beta)$.

This follows directly from 56, in case the collection π of that theorem reduces to the pair (x, y).

64. If α, δ, ϵ are collections, and b and c are elements, such that (1) $F(b\alpha|\delta)$ and (2) $F(c\alpha|\delta)$, while (3) $F(bc|\epsilon)$, then $F(\delta|\alpha\epsilon)$.

For by (1) $F(b\alpha|\delta)$ while by (2), $F(\alpha|\delta\bar{c})$. Let $b = y$, and $c = \bar{e}$. Then $F(y\alpha|\delta)$, while $F(\alpha|\delta e)$; and, by (3), $F(y|\epsilon e)$, i. e., $F(e|y\bar{\epsilon})$.

Hence, by 62, $F(\bar{\epsilon}\delta|\alpha)$. Whence follows $F(\alpha\epsilon|\delta)$. That is, if two collections α and δ are such that if either b or c be separately adjoined to α, the resulting collection forms an F-collection with δ, and if the pair (b, c) forms an F-collection with ϵ, then if ϵ itself be adjoined to α, the resulting collection forms an F-collection with δ.

Chapter IV. F-triads, mediators and antecedents.

65. What elements exist in the system Σ we have as yet but very imperfectly investigated. Yet before we proceed to this investigation, it will prove convenient to outline the general character of the order which is possible in the system Σ, so far as we have yet developed this order. The fact that given elements do or do not belong to a certain O-collection, or do or do not constitute an O-collection, is one which appears directly to establish no sort of order amongst the elements of Σ. The relation in which various elements stand to one another when they belong to the same O-collection, is so far absolutely *symmetrical*, and nothing can be said of one member of such a collection which is not asserted of everyone of the others, so far as this collection is concerned.

But the fact that every element of Σ possesses an obverse, enables one to establish relations between certain elements, or sets of elements, relations which are due to the further fact that given elements may enter into O-collections with the *obverses* of certain other elements. The consequence of this is that F-collections are definable. Let us provisionally assume that a large variety of F-collections exist in Σ.

66. F-collections, if indeterminate, are, like O-collections, of a wholly symmetrical structure, and their members are so far undistinguished from one another. But the members of a determinate F-collection are no longer so symmetrically disposed. It is indeed true that the order in which the elements in the set of an F-collection on each side of the vertical are considered, is indifferent. Nor does it make any difference which set is written before or after the vertical. But if $F(\beta|\gamma)$, then, in general, any element x of β is related to any element y of γ in a way which is *not* reciprocated. For x is related to y as that element which, in combination with some collection β' of companion elements, forms an F-collection with y, when y is combined with a collection γ' of elements (β' being the collection of the other elements of β besides x, γ' being the collection of the other elements of γ besides y). This relation, if read in the other direction, changes, in general, its character, and so is an unsymmetrical relation.

But, as 59 has shown us, this unsymmetrical dyadic relationship is *transitive.* In terms of this relation certain sets of the elements may be *ordered* and so arranged in series like points on a line.

67. The fundamental form of such series becomes manifest if we pay attention to those cases of the much more general theorems regarding F-collections, which appear as special results if we consider only *triads* of elements.

In this case if, for example, $F(ac|b)$ or $F(ca|b)$, that is, if $O(ac\bar{b})$, we shall call b "the *mediator* between a and c," or, where that is more convenient, the "mediator of the pair (a, c)." The "mediator" of a pair is accordingly an element whose obverse forms an O-triad when adjoined to that pair. If $O(pqr)$, then $F(pq|\bar{r})$, so that the obverse of any member of an O-triad is the mediator of the pair formed by the other elements of that triad.

The relation of the mediator to the elements which it mediates may be treated at pleasure either as a triadic or as a dyadic relation. In order to treat it as a dyadic relation we may take account of the fact that, if $F(ac|b)$, b is in a certain relation to *a with respect to* c, and is in a certain relation to c *with respect to* a. This aspect of the matter may become especially important in case we deal with a number of triads in which the mediators are any elements whatever, while all the pairs mediated have a common element y. Such pairs appear if $F(ym|q)$, $F(yn|r)$, $F(yo|s)$, etc. In all such cases q has the same relation to m that r has to n, and that s has to o, etc., since q is in a given relation to *m with respect to* y; and the same holds true in the other instances in question.

68. If $F(yq|p)$, we may, whenever that is convenient, first select one element of the pair (q, y), say the element y, and thereupon say that p is in a relation

to q which we shall call the relation: "*antecedent of q with respect to y.*" The element y we may hereupon call the "*origin*" from which the relation is defined or reckoned. Equally, if we choose the element q as our origin, we can say that p is in the relation to y of being "an antecedent of y with respect to q". We shall symbolize the assertion "p is antecedent of q with respect to y," by the expression $p \prec_y q$ or $q\ {}_y\!\succ p$; and we can also at pleasure read either of these expressions thus: "q is a consequent of p with respect to y." This expression means precisely the same as the expression $F(yq|p)$, or as the expression $O(yq\bar{p})$. Expressing the facts in the new way, as involving the relation of an antecedent to a consequent, has merely the advantage of bringing out certain aspects of the situation which will be conveniently expressible in terms of a dyadic relation — a relation which, as already pointed out, will prove to be unsymmetrical and transitive and, therefore, useful for the definition of serial order amongst certain specially selected elements of Σ. The equivalence of meaning of the three expressions: $O(qy\bar{p})$, $F(qy|p)$, and $p \prec_y q$, enables us at once to see how superficial is the difference between symmetrical and unsymmetrical relations. All that any one of these three expressions asserts is that the two perfectly symmetrical O-collections $O(qyx)$ and $O(xp)$ (where $x = \bar{p}$), both exist as collections of the elements of Σ.

69. It is of course expressly true that the relation of antecedent to consequent, or of consequent to antecedent, has meaning *only* with reference to a given origin. It is this origin which gives "*sense*" to the pair (pq) in the expression $p \prec_y q$. On the other hand, if a question arises as to whether the "sense," or asymmetry, of the dyadic relation here in question, is a fundamental fact, or is unanalyzable — this question is answered in advance by our derivation of the whole asymmetry from the perfectly symmetrical properties which characterize the various members of any O-collection.

The system Σ, as we shall hereafter see, includes elements whose relations are precisely the ones which are of the most fundamental importance in all the exact sciences. The customary procedure of these sciences may be said, in the main, to involve the definition of these relations in terms of the relation of antecedent and consequent. Wherever a linear series is in question, wherever an origin of coördinates is employed, wherever "cause and effect," "ground and consequence," orientation in space or direction of tendency in time are in question, the dyadic asymmetrical relations involved are essentially the same as the relation here symbolized by $p \prec_y q$.

This expression, then, is due to certain of our best established practical instincts and to some of our best fixed intellectual habits. Yet it is not the only expression for the relations involved. It is in several respects inferior to the more direct expression in terms of O-relations. The range of its efficacy as an expression will become clearer hereafter. When, in fact, we attempt to describe the relations of the system Σ merely in terms of the antecedent-consequent relation, we not only limit ourselves to an arbitrary choice of origin, but miss the power to survey at a glance relations of more than a dyadic, or triadic character.

Properties of the relation of mediator, antecedent, and consequent.

70. If $F(ab|c)$, and $a=b$, then $c=a=b$. For this is a special case of the principle proved in 55.* Furthermore, if $p \prec_y q$ and $q \prec_y p$, then $p=q$, since this is but another expression of 52 (2). Consequently, if $p \neq q$, and $p \prec_y q$, then $q \prec_y p$ is false. The relation of antecedent to consequent, if it obtains at all between a pair of non-equivalent elements, is therefore in that case inconvertible; hence the relation becomes totally asymmetrical so soon as it is confined to pairs of non-equivalent elements.

71. If $F(a|by)$ and $F(b|cy)$, then $F(a|cy)$ and $F(b|ac)$.

If $a \prec_y b$ and $b \prec_y c$, then $a \prec_y c$; and $b \prec_a c$.

The proof is as in 59. Thus from $F(b|a\bar{y})$, and $F(b|cy)$ follows $F(a\bar{y}|cy)$ and $F(a|cy)$. From $F(y|a\bar{b})$ and $F(y|b\bar{c})$ follows $F(a\bar{b}|b\bar{c})$; and consequently $F(b|ac)$. The form of the theorem stated on the right is simply a direct translation into the symbolism of the relation of antecedent and consequent. The latter relation is thus shown to be transitive.

72. If $F(x|ab)$ and $F(x|bc)$, while $F(b|ac)$, then $x=b$. For since $F(x|bc)$ while $F(b|ca)$, we have $F(c|x\bar{b})$ and $F(c|b\bar{a})$. Whence follows $F(x|b\bar{a})$; that is $F(b|ax)$. From $F(x|ab)$ and $F(b|ax)$ follows, by 52 (2) $x=b$.

In other words, if $x \neq b$, and b is a mediator of the pair (a, c), it is impossible that x should be at once a mediator of the two pairs (a, b) and (b, c).

73. If $F(x|ab)$ and $F(y|ab)$ and $F(d|xy)$, then $F(d|ab)$.

If $x \prec_a b$ and $y \prec_a b$ and also $d \prec_x y$, then $d \prec_a b$.

This follows directly from 63 and expresses the fact that a mediator of two elements which, with respect to a given origin, are antecedents of the same element, is itself an antecedent of that element with respect to the same origin.

74. If $F(yb|d)$, $F(yc|e)$ and $F(ed|b)$, then $F(cd|b)$.

If $d \prec_y b$ and $e \prec_y c$, while $b \prec_d e$, then $b \prec_d c$.

This is a special case of the proposition proved in 62. If two pairs, (b, y) and (c, y), have a common element y, and if each pair forms a determinate F-triad when a term d (in one case), or e (in the other case) is set on the opposite side of the vertical, and if the member b of the one pair is a mediator of e and d, while d is the third member of the F-triad in which b occurs, then b is a mediator of d and of c, where c is the remaining member of the other pair (c, y).

* This theorem is used by KEMPE as a fundamental principle in defining the relation here called that of Mediator.

Otherwise: If, with respect to a common origin y, the elements d and e are, respectively, antecedents of the elements b and c, so that d is antecedent of b and e of c, then, if b is a mediator of d and e, b is also a mediator of c and d.

75. If $F(ab|d)$, $F(ac|d)$ and $F(bc|e)$, then $F(ae|d)$.

If $d \prec_a b$ and $d \prec_a c$ and $e \prec_b c$, then $d \prec_a e$.

This follows from 64.

That is, whatever element is mediator of two pairs which have an element in common, is mediator of the pair composed of the common element and of any mediator of the pair, formed by the elements which, belonging to the original pairs are not common to them. Otherwise, whatever element is, with respect to a given origin, an antecedent of each of a pair of elements, is antecedent of any mediator of this pair.

76. Any element a is a mediator between any element y and the obverse of y; and any element y is a mediator between itself and any other element a.

For $O(\bar{a}y\bar{y})$ and $O(ay\bar{y})$.

77. For any origin y, it is true, that y is an antecedent of itself and of every other element, including $\bar{y}$; while any element a is an antecedent of itself and of $\bar{y}$.

For, $O(\bar{y}ya)$, $O(\bar{y}yy)$ and $O(\bar{y}y\bar{y})$, while $O(\bar{a}ay)$ and $O(\bar{a}y\bar{y})$.

78. If $a \prec_y b$, then $b \prec_y \bar{a}$.

For $O(\bar{a}by)$, and hence $F(\bar{\bar{b}}|\bar{a}y)$.

79. Whatever element is, with respect to a given origin y, an antecedent of every member of an O-collection except one, is also an antecedent of the obverse of this excepted element.

Let π be any collection, and e such an element that $O(\pi e)$. Then, by this hypothesis $F(\pi|\bar{e})$. Let q be an element such that $F(q|x_r y)$ is true of every element x_r of π so that $q \prec_y x_r$. Then, by transfer, $F(q\bar{y}|x_r)$. Hence the collection $(q, \bar{y})$ forms a determinate F-collection, when set on one side of the vertical, with *any* element of the determinate F-collection $F(\pi|\bar{e})$ on the other side of the vertical, except the element $\bar{e}$. Hence, by 57, it is *also* true that $F(q\bar{y}|\bar{e})$. Hence $F(q|y\bar{e})$. Hence $q \prec_y \bar{e}$.

80. If, with respect to y, q is an antecedent of *every* member of a given O-collection, then q is equivalent to y. For q is in any case an antecedent, with respect to the origin y, of every member of the collection except any member e. Hence $q \prec_y \bar{e}$. But, by the present hypothesis, $q \prec_y e$ is *also* true. Hence $O(\bar{q}ye)$ and $O(\bar{q}y\bar{e})$. Hence $O(\bar{q}y)$. Hence $q = y$.

81. Whatever element is, with respect to a given origin, a consequent of every member of an O-collection except one, is also a consequent of the obverse of this excepted element.

Suppose π such that $F(\pi|\bar{e})$ as before. And suppose that, for every element x_r of π, $x_r \prec_y q$, so that $F(x_r|yq)$. By the same reasoning as that of the last theorem $F(\bar{e}|qy)$; and hence $\bar{e} \prec_y q$.

82. If q is a consequent of every member of the collection, then it is also true that $e \prec_y q$; whence $O(qye)$ and $O(qy\bar{e})$. Hence $O(yq)$, and consequently $q = \bar{y}$.

Chapter V. The existence of elements. chains. resultants. conjugate pairs of resultants. the algebra of logic.

83. In discussing, in the foregoing, the most characteristic relations of the elements of Σ, we have tacitly and provisionally assumed that elements may exist in sufficient variety to exemplify these relations. Our existential principles have so far been used mainly to establish the existence of pairs of obverse and of non-equivalent elements. We must now proceed to survey, more in detail, the actual structure of the system Σ.

84. By principles III and IV, the system Σ contains a pair of non-equivalent elements, say (xy). By 26, the pair $(\bar{x}, \bar{y})$ also exists. It is however in so far possible that $x = \bar{y}$, and hence that $y = \bar{x}$. In that case the members of the pair (x, y) become respectively equivalent to the members of the pair $(\bar{y}, \bar{x})$, or, again, to the members of the pair $(x, \bar{x})$. Were the system Σ to consist *merely* of the single pair of mutually non-equivalent and mutually obverse elements, $(x, \bar{x})$, principles I, II, III, IV, and VI would all of them be true of the system as thus restricted. For $O(x\bar{x})$; and if, to this collection, we add any collection γ, consisting either of x, or of $\bar{x}$, repeated any multitude of times, or again consisting both of x and of $\bar{x}$, in any combination, each occuring any multitude of times — in any case $O(x\bar{x}\gamma)$. Hence principle I is satisfied. On the other hand since $O(x\bar{x})$, while, by 35, 37, $E(xx)$, and $E(\bar{x}\bar{x})$ — the non-equivalence of x and $\bar{x}$ can be readily established, without taking account of any collections except those into which x and $\bar{x}$ either jointly or severally enter. In order that a collection β shall consist altogether of complements of δ, while $O(\beta)$ is true, it is necessary, in case Σ contains *only* x and $\bar{x}$, that β should include both x and $\bar{x}$; while δ (which, by hypothesis is such that $O(\delta b_n)$ for every element b_n, of β), must then *also* include both x and $\bar{x}$. Hence if $O(\beta)$, $O(\delta)$ follows. Hence principle II would hold true if the system Σ consisted only of the pair $(x, \bar{x})$. Principles III and IV would obviously hold true of the same system. And considered with reference to x, its obverse, $\bar{x}$, is an element satisfying the requirements of principle VI; while the same holds true of x when it is considered with reference to $\bar{x}$. All the principles except V would therefore be satisfied if the system Σ consisted of the single pair $(x, \bar{x})$.

Chains of elements, defined by recurrence.

85. Principle V, however, is not satisfied by the existence of a single pair of elements, such as $(x, \bar{x})$. For, if the pair $(x, \bar{x})$ exists, then (since $x \mathrel{+\!\!\!|} \bar{x}$ by

38), it follows that principle V demands the existence of r_1 such that $E(r_1 x)$ and $E(r_1 \bar{x})$. These two assertions require that $r_1 \neq \bar{x}$ and $r_1 \neq x$ (by 41). Since $r_1 \neq \bar{x}$, there exists an element which we will next symbolize by $\bar{r}_2$, and which, by virtue of principle V, is such that $E(r_1 \bar{r}_2)$, $E(\bar{r}_2 \bar{x})$, and $O(r_1 \bar{r}_2 \bar{x})$. The obverse of $\bar{r}_2$ also exists. Since $E(r_1 \bar{r}_2)$ it follows by 41, that $r_2 \neq r_1$; and for the precisely analogous reason, $r_2 \neq \bar{x}$. At the same time $F(r_2 | r_1 \bar{x})$. Since $r_2 \neq \bar{x}$, there also exists, by principle V, an element $\bar{r}_3$, such that $E(\bar{r}_3 r_2)$ and $E(\bar{r}_3 \bar{x})$ while $O(r_2 \bar{r}_3 \bar{x})$. The obverse r_3 of $\bar{r}_3$, therefore, also exists; and is such that $r_3 \neq r_2$, $r_3 \neq \bar{x}$, and $F(r_3 | r_2 \bar{x})$. The procedure whereby the elements r_1, r_2, and r_3 have been defined is obviously a recurrent one. Repeatedly applied it defines a chain of elements: $(r_1, r_2, \cdots, r_n, \cdots)$, whereof any finite number n may be at pleasure defined in a determinate order. These elements, together with their obverses $(\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n, \cdots)$, all exist in Σ, and possess the following properties:

(1) $$x \neq r_1;\ r_1 \neq r_2;\ r_2 \neq r_3;\ \cdots;\ r_{n-1} \neq r_n;\ r_n \neq \bar{x}.$$

(2) $$F(r_1 | x\bar{x}),\ F(r_2 | r_1 \bar{x}),\ F(r_3 | r_2 \bar{x}),\ \cdots,\ F(r_n | r_{n-1} \bar{x}).$$

(3) $$F(\bar{r}_1 | x\bar{x}),\ F(\bar{r}_2 | \bar{r}_1 x),\ F(\bar{r}_3 | \bar{r}_2 x),\ \cdots,\ F(\bar{r}_n | \bar{r}_{n-1} x).$$

(4) $$\bar{x} \neq \bar{r}_1;\ \bar{r}_1 \neq \bar{r}_2;\ \bar{r}_2 \neq \bar{r}_3;\ \cdots;\ \bar{r}_{n-1} \neq \bar{r}_n;\ \bar{r}_n \neq x.$$

The expressions (3) and (4) follow directly from the truth of (1) and of (2), by the definitions of the F-collections, and of the relations of equivalent and of obverse elements, and by 30, 36(2), and 52(4).

From these conditions it further follows that, if m and n are (for the moment) viewed, not as symbols for elements of Σ, but as purely numerical marks or indices, serving to distinguish the ordinal positions of different members of the chain $(r_1, r_2, \cdots, r_m, \cdots, r_n, \cdots)$, and if m, in the series of natural numbers, precedes n, then:

(5) $$F(r_n | r_m \bar{x});\ \text{and } r_n \neq r_m.$$

(6) $$F(\bar{r}_n | \bar{r}_m x);\ \text{and } \bar{r}_n \neq \bar{r}_m.$$

For, by the laws of the construction of the chain, $F(r_{m+1} | r_m \bar{x})$, and $F(r_{m+2} | r_{m+1} \bar{x})$. Hence $F(r_{m+2} | r_m \bar{x})$, by 71. In the same way, since $F(r_{m+3} | r_{m+2} \bar{x})$, there follows $F(r_{m+3} | r_m \bar{x})$. And the same process of elimination can be repeated any number of times, so that, in fine, $F(r_n | r_m \bar{x})$. Meanwhile, by the principle proved in 72, $r_{m+2} \neq r_m$. In fact, if we suppose $r_{m+2} = r_m$, we have $F(r_{m+1} | r_m \bar{x})$ by construction, as well as $F(r_{m+2} | r_{m+1} \bar{x})$. By the substitution of equivalents we therefore obtain:

$$F(r_{m+1} | r_m \bar{x}) \qquad \text{and} \qquad F(r_m | r_{m+1} \bar{x}).$$

Whence follows, by 52 (2), $r_{m+1} = r_m$, which contradicts the conditions stated in the expressions (1). Hence $r_{m+2} \neq r_m$. In the same way we can prove that $r_{m+3} \neq r_m$; and in general, $r_n \neq r_m$, where n is any ordinal number that follows m. Hence the expressions (5) are proved true; and the expressions (6) follow therefrom by considering the obverses of the elements occurring in (5). All the members of the chain $(r_1, r_2, \cdots, r_n, \cdots)$ are such that $F(r_n | r_1 \bar{x})$.

To sum up: The requirements of principle V include the assertion that, since the pair $(x, \bar{x})$ exists there must also exist the two distinct chains of elements:

$$(r_1, r_2, \cdots, r_n, \cdots) \qquad \text{and} \qquad (\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n, \cdots).$$

No two of the elements of either one of these chains are mutually equivalent. If any element of the first chain, as r_m, were equivalent to a member of the second chain, say $\bar{r}_n$, we should have, since, by the foregoing:

$$F(r_m | r_1 \bar{x}) \qquad \text{and} \qquad F(\bar{r}_n | \bar{r}_1 x),$$

the consequence: $F(r_m | r_1 \bar{x})$ and $F(r_m | \bar{r}_1 x)$. By elimination would follow $F(r_1 \bar{x} | \bar{r}_1 x)$; whence would follow $O(r_1 x)$ and so $r_1 = x$, a result which is rendered impossible by the conditions that define r_1. Hence no element of either chain can be equivalent to any member either of the same chain, or of the other chain; and all the elements of both chains are non-equivalent both to x and to $\bar{x}$. Since, if any element r_n exists in the first chain, r_{n+1} also exists (while $\bar{r}_{n+1}$ also belongs to the other chain), the two chains contain each an infinite number of non-equivalent elements. *The system Σ consequently contains an infinite number of mutually non-equivalent elements.*

86. Herewith, however, the requirements of principle V are by no means exhausted. For since no two elements of either chain are mutually equivalent, any two successive elements of each chain, as, for instance, r_m and r_{m+1}, are such that, by principle V, $\bar{s}_1$ exists such that $E(r_m \bar{s}_1)$ and $E(r_{m+1} \bar{s}_1)$, while $O(r_m r_{m+1} \bar{s}_1)$. In this case $F(s_1 | r_m r_{m+1})$, while $s_1 \neq r_m$ and $s_1 \neq r_{m+1}$. A new recurrent process is thus defined, a process which can be employed to define a chain $(s_1, s_2, \cdots, s_j, \cdots)$ where the subscripts again have the significance of the ordinal numbers.

By 72, no two elements of this chain can be mutually equivalent. Every member s_j of this new chain must be non-equivalent to any member of the chain $(r_1, r_2, \cdots, r_m, \cdots)$, as well as to any member of any chain $(s'_1, s'_2 \cdots, s'_k, \cdots)$, or $(t_1, t_2, \cdots, t_h, \cdots)$, such as can be formed by taking account of those pairs: (r_1, r_2), (r_3, r_4), etc., which are different from the pair (r_{m+1}, r_m), and by applying principle V recurrently to them. In general, if $F(e | r_m r_{m+1})$, and $F(g | r_n r_{n+1})$, where m and n are different subscript numbers, while m precedes n in the ordinal series, we can express the relations already considered by taking, if we choose, r_1 as an origin, and by employing the relation of antecedent and consequent. In this case we can write, upon the basis of 85 (5),

$$r_m \prec_{r_1} r_{m+1},$$

$$F(e \mid r_m r_{m+1});\ e \mp r_m;\ e \mp r_{m+1}.$$

Whence follows $e \prec_{r_1} r_{m+1}$.

Furthermore: $r_{m+1} \prec_{r_1} r_n$ by 85 (5),

$$r_n \prec_{r_1} g,\ r_n \mp g.$$

And so, if $e = g$, we have:

$$g \prec_{r_1} r_{m+1} \prec_{r_1} r_n \prec_{r_1} g.$$

This, however, is impossible. Thus no member of any one of the new chains is equivalent to any member of any other of these new chains, or of the original chains. Since an infinite number of different pairs of non-equivalent members exist in the chain $(r_1, r_2, \cdots, r_n, \cdots)$, it follows that an infinite number of new chains can be constructed upon the basis of these various pairs. The process of forming such chains is itself recurrent.

In addition to the chains thus far defined, other chains of elements exist in Σ. For since, by 85, $r_1 \mp x$, it is possible to treat the pair (r_1, x) as, in 85, the pair $(r_1, \bar{x})$ was treated. The result would be to define a chain of elements $(r_1, p_1, p_2, \cdots, p_h, \cdots)$, such that $F(p_h \mid r_1 x)$, while, as before, no two elements of the chain are mutually equivalent. If any element of Σ, say v, is such that $F(v \mid r_1 x)$ and $F(v \mid r_1 \bar{x})$, we have $F(v \mid r_1 x)$ and $F(r_1 \mid vx)$, and hence $v = r_1$. See also 72. It follows that no element of the new chain except r_1 itself is equivalent to any element of the chain $(r_1, r_2, \cdots, r_n, \cdots)$.

General properties of resultants.

87. A very little consideration serves to show that a new application of principle VI, to the chains of elements now defined, will lead to still further results. Before we are prepared to consider these results, we must however survey the properties of a class of elements defined in 17. Of the infinitely numerous non-equivalent elements now known to exist in Σ, collections can be made comprising any number of elements. These may be either E-collections or O-collections. If of the former type, the collections so made can be enlarged to O-collections by the adjunction of suitable elements. To what laws are such adjunctions subject? This we are next to see.

88. If β be any collection, any element r such that $F(r \mid \beta)$, is a resultant of the collection β, by virtue of the definition stated in 17. Since r is thus any element such that $O(\beta\bar{r})$, and since the obverse of any element of β may fill the place of $\bar{r}$ in this O-collection, it is obvious that every element of β is also a resultant of β. Furthermore, if any element x forms an O-collection with any partial collection of elements of β, $\bar{x}$ is a resultant of β. For if $O(\lambda x)$, then $O(\kappa\lambda x)$, so that if (κ, λ) is the same collection as β, $F(\beta \mid \bar{x})$.

89. If $F(r|\beta)$, $F(t|\beta)$ and $F(u|rt)$ are all true there follows, by 63, $F(u|\beta)$. Hence any mediator of a pair of resultants of a given collection is itself a resultant of the collection. By the reasoning used in 85–86, it can therefore be shown that any collection containing at least two non-equivalent elements must possess an infinite number of resultants.

90. If ρ be the collection consisting of all the resultants, r_1, r_2, etc., of a given collection β, then any resultant of ρ is also a resultant of β. .

Let q be such that $F(q|\rho)$. Then $O(\rho\bar{q})$, and hence $O(\bar{\rho}q)$. By the definition of a resultant, β is a collection which becomes an O-collection if any element of $\bar{\rho}$, that is, if the obverse of any one of the resultants of β, be adjoined to β. Hence β forms an O-collection in case any member of the O-collection $O(\bar{\rho}q)$, except q, is adjoined to β. By 36(5) it follows that $O(\beta\bar{q})$. Hence $F(\beta|q)$.

The collection ρ is consequently a collection which contains all of its own resultants. It also contains the resultants of all the partial collections which can be found by selecting certain elements from the collection β.

91. If y is an element of Σ, selected at pleasure, and if β is any collection, and if a determinate element c exists such that, whatever element b_n of β is chosen, c is a mediator of the pair (y, b_n), then c is also a mediator of any and every pair that can be formed by combining y with any resultant whatever, say r_x of β. For if r_x be any resultant of β, then $O(\beta\bar{r}_x)$. And if c exists such that $F(c|yb_n)$ is true, whatever element b_n of β may be chosen, then $O(\bar{c}yb_n)$ is true of every member b_n of β. Hence, by 36(5), $O(\bar{c}yr_x)$. And hence $F(c|yr_x)$.

The converse of this theorem is obvious. That is, if an element c is a mediator of *every* pair (y, r_x) consisting of y and some resultant of β, c is a mediator of (y, b_n) where b_n is any element chosen at pleasure from β. For the elements of β are themselves amongst the resultants of β.

There always exists an element c having the properties here in question, since y itself is such an element. For $F(y|yx)$ is true of every element x. But, as will soon appear, there are, in certain important classes of cases, elements possessing this property which are not equivalent to the chosen element y.

92. If γ be the collection of all those elements, c_v, which possess the property discussed in 91, viz., if γ be the collection of those elements c_v, such that $F(c_v|yr_x)$ is true for a determinate selected element y, and for every resultant, r_x, of β, then every resultant of γ is itself a member of the collection γ.

For let t be such that $F(t|\gamma)$, so that $O(\bar{\gamma}t)$. Since (y, r_x) is such that $O(yr_x\bar{c}_v)$ is true of every element $\bar{c}_v$ of $\bar{\gamma}$, while $O(\bar{\gamma}t)$, it follows that $O(yr_x\bar{t})$, by 36(5). Hence $F(t|yr_x)$.

If ρ be the collection of all the resultants of β, and if γ be the collection of all those elements, such as c_v, which with reference to a selected element y, have the property of being, every one of them, such that $F(c_v|yr_x)$ is true of every element r_x of ρ, then, as now appears, the collections γ and ρ are collections each of which contains all of its own resultants.

Pairs of conjugate resultants.

93. We are now prepared to consider more minutely the consequences of principle VI. By that principle, if a collection ϑ possesses any complement w, so that $O(\vartheta w)$, a complement v also exists such that $O(\vartheta v)$ while, whatever element t_n of ϑ be selected, $O(vwt_n)$. Stating this requirement in terms of the F-collections, we have, as the conditions set forth in the hypothesis of the principle, the existence of an element $\bar{w}$, such that $F(\bar{w}|\vartheta)$. The consequence according to the principle is that $\bar{v}$ also exists such that $F(\bar{v}|\vartheta)$, while, since $O(vwt_n)$, $O(\bar{v}\bar{w}\bar{t}_n)$, and therefore $F(t_n|\bar{v}\bar{w})$ for every element t_n of ϑ. Let $\bar{w}=q$, and $\bar{v}=r$. Then principle VI asserts that whatever resultant q of a collection ϑ be selected, there always exists a resultant of ϑ, namely r, such that every element of ϑ is a mediator (that is a resultant) of the pair (q, r). It readily follows that every resultant of ϑ is a mediator of (q, r). For if p_k be any resultant of ϑ, then $O(\vartheta\bar{p}_k)$. And since $O(vwt_n)$ is true of every element of ϑ, while $O(\vartheta\bar{p}_k)$, it follows, by 36 (5), that $O(vwp_k)$, that is $O(\bar{q}\bar{r}p_k)$, and hence $F(p_k|qr)$.

94. If q is any determinate resultant of ϑ, and if r is an element related to q in the way set forth in 93, then any element, r', such that $F(qr'|p_k)$ is true for every resultant of ϑ, i. e. any element r' such that $F(qr'|t_n)$ is true of every element of ϑ, is also such that $r'=r$. For, since r and r' are both of them resultants of ϑ, we have $F(qr|r')$, because of the definition of r; and also $F(qr'|r)$ because of the definition of r'. Hence $r'=r$ by 52 (2). We may consequently let r stand as the unique representative of the class of those resultants of ϑ which, when q is given, fulfil, with respect to q, the requirement of principle VI. With this understanding, we shall henceforth characterize r as the *conjugate resultant of q in, or with respect to* the collection ϑ. If r is given instead of q, some equivalent of q is nevertheless predetermined as a conjugate of r; and if q be selected as the unique representative of the class of elements which are equivalent to itself, we may regard the relation of q and r as wholly symmetrical; and so we may henceforth speak of the pair (q, r) as a *pair of conjugate resultants of the collection* ϑ, or more briefly as a *conjugate pair in, or with respect to,* ϑ. We shall symbolize the relation in question thus: $J(qr;\ \vartheta)$. This symbol is to be read as the assertion: "The pair (q, r) is a pair of conjugate resultants of the collection ϑ," or "is a conjugate pair in ϑ," or "with respect to ϑ." Were ϑ a pair, as for instance (x, y), we could write $J[qr;\ (x, y)]$.

95. If the collection ϑ is made to include all of the elements of Σ, the pair (q, r) becomes a pair of mutually obverse elements so that $q = \bar{r}$. This appeared already in 27, when the first use of principle VI was made. If ϑ is an O-collection, a conjugate pair (q, r) such that $O(\bar{q}\bar{r}t_n)$ while $O(\vartheta)$ is true, is a pair such, by principle II, that $O(\bar{q}\bar{r})$. Hence, in this case, $O(qr)$, and thus any pair that is a pair of conjugates with respect to an O-collection is an O-pair. If, in case of any collection ϑ, a pair (q, r), conjugate with respect to ϑ is such that $q = r$, then, since $F(t_n | qr)$, every element t_n of ϑ is such that $t_n = q = r$; and therefore, in this case, a single element may be taken as the unique representative both of the whole collection ϑ, and of all of its possible pairs of conjugate resultants (see 52 (1), and 70).

96. In case of any pair of conjugate resultants of a system β, i. e., in case of (q, r) such that $J(qr; \beta)$, the resultants of the pair (q, r), and the resultants of β, form precisely identical collections. If β is enlarged either to an O-collection, or so as to include all of the elements of Σ, the entire system Σ becomes the collection of the resultants of any one of the possible pairs of obverse elements of Σ, such, for instance as $(x, \bar{x})$; any one of these pairs being, as we now know, such that, if $O(\beta)$, $J(x\bar{x}; \beta)$.

97. If β is a given collection, and if (q, r) is a conjugate pair of its resultants, and if u is any third resultant of β, not equivalent either to q or to r, then the conjugate resultant v of u can be found by considering merely the triad (q, r, u). According to principle VI there is, namely, a resultant v of this triad* such that $F(vu | q)$ and $F(vu | r)$. Since v is a resultant of the triad, $F(qru | v)$. But meanwhile, since $J(qr; \beta)$, and $F(\beta | v)$, $F(qr | v)$ by the definition of a conjugate pair, so that from $F(qru | v)$ the element u may be stricken out. Since v, then, is such that $F(qr | v)$, while $F(vu | q)$ and $F(vu | r)$, it is easy to show that any resultant p_k of β is such that $F(p_k | vu)$. For $F(p_k | qr)$, since (q, r) is a conjugate pair. But from $F(p_k | qr)$, $F(q | uv)$ and $F(r | uv)$, follows, by 73, $F(p_k | uv)$.

If, then, a single conjugate pair of resultants of a collection β is given, viz., (q, r), the conjugate of any third resultant of β, such as u, is equivalent to the conjugate of u in the collection of the resultants of the triad (q, r, u). If β is an O-collection, or if β includes all of the elements of Σ, the conjugate of x in the triad $(x, z, \bar{z})$ is obviously $\bar{x}$.

98. If (β, x) be any collection that includes a given element, x, the conjugate resultant of x with respect to the collection (β, x) is one of the resultants of β. For let q be such a resultant. $F(q | qx)$ is in any case true of q. But q has, in addition, to be such that $F(q | \beta x)$, while, whatever element b_n, of β,

* Identical with what KEMPE calls the "unsymmetrical resultant" of the triad (q, r, u). KEMPE does not directly define our conjugate resultants in general, but builds his theory upon that of the resultants of triads.

be selected, $F(qx|b_n)$; i. e., $O(\bar{q}\bar{x}b_n)$. By the adjunction of all elements of β besides b_n, we obtain, hereupon, $O(\bar{q}\bar{x}\beta)$. But since $F(q|\beta x)$, $O(\bar{q}x\beta)$. Hence $O(\bar{q}\beta)$. Hence $F(\beta|q)$.

99. If β and α are such that $F(\beta|\alpha)$, then β and α possess at least one common resultant. For let a be an element of α, chosen at pleasure. Adjoin $\bar{a}$ to the collection β, and consider the conjugate resultant of $\bar{a}$ in the collection $(\bar{a}, \beta)$. Let this resultant be the element b. By 98, $F(\beta|b)$. By the definition of a conjugate resultant $F(\bar{a}b|b_n)$ is true of every element b_n of β. By hypothesis, however, $F(\beta|\alpha)$. Hence by 56, $F(\alpha|\bar{a}b)$. Since a is itself an element of α, the superfluous obverse $\bar{a}$ may be omitted, so that $F(\alpha|b)$. Hence the element b is a resultant of β and also of α.

The importance of this theorem for the geometrical application of our theory (since the theorem may be called *the theorem regarding intersections* or *transversals*) justifies a proof directly in terms of O-collections.

If, namely, $O(\delta\epsilon)$, there exists x such that $O(\delta\bar{x})$ while $O(\epsilon x)$. For let us select at pleasure any element d of δ. Consider the collections $(\bar{d}, \bar{\epsilon})$, and, with respect to that collection, define the conjugate resultant of $\bar{d}$ in $(\bar{d}, \bar{\epsilon})$. Let x be this resultant. By 98, $O(\bar{\epsilon}\bar{x})$ is true. Hence $O(\epsilon x)$. By the definition of a conjugate resultant $F(\bar{e}_r|x\bar{d})$ is true of every element $\bar{e}_r$ of $\bar{\epsilon}$. Hence $O(e_r x\bar{d})$ is true of every element e_r of ϵ, while $O(\delta\epsilon)$. Hence, by 24 (4), $O(\bar{\delta}x\bar{d})$. $\bar{d}$ is a repetition of some element of $\bar{\delta}$, and may be stricken out (by 37). Hence $O(\bar{\delta}x)$. Hence $O(\delta\bar{x})$, while, as above shown, $O(\epsilon x)$. So the theorem is proved.

Since this process may be repeated for every element of δ and also of ϵ, the variety of elements of the type x, in case the elements of δ and of ϵ include non-equivalent pairs, is, in general, by principle V, and 89, infinite.

The relations of pairs of conjugate resultants in various collections.

100. It is frequently important to bring the various pairs of conjugate resultants which exist in different collections into relation with one another. The procedure by which this is accomplished will lead us at once to the threshold of the ordinary algebra of logic, which, as originally developed, was based upon observing certain properties of the system Σ, in cases where this system was interpreted as a collection whose elements are either classes or propositions.

Let the conjugate resultant of y in the collection (y, β) be x, so that $J(xy; (\beta, y))$. Then, by 98, $F(x|\beta)$ is true. Hereupon, if we select, amongst the resultants of β, that one, say z, which is the conjugate resultant, with respect to β, of the element x just determined, so that, while $J(xy; (\beta, y))$, it is also true that $J(xz; \beta)$, then it follows that $F(yb_n|z)$ is true for every element b_n of β. For z is such that $F(xz|b_n)$, and is also such that $F(\beta|z)$. Since $F(b_n|xy)$ is true (by the foregoing) for every element b_n of β, while

$F(\beta|z)$, it follows, by 36 (5) and 57, that $F(xy|z)$; and since $F(b_n|xz)$, that is $F(b_n\bar{z}|x)$, while $F(x|z\bar{y})$, we have $F(b_n\bar{z}|z\bar{y})$, whence follows $F(z|b_ny)$.

We have, therefore, the result, that, if y be any element, and β any collection, then, in case x is such that $J(xy;\ (\beta, y))$ and z is such that $J(xz;\ \beta)$, the two elements x and z constitute a conjugate pair of the resultants of β, while just this pair stands in what we may regard as an unique relation to y. The pair (x, z) is namely such that, for the first, z is a mediator between y and any element b_n of β which may have been chosen for comparison with y and with z. By 91, z is consequently also a mediator between y and any resultant, r_v, of β, so that, whatever resultant r_v, of β, we may select $F(r_vy|z)$. Moreover, whatever resultant z' of β possesses the property just ascribed to z, must be equivalent to z. For if z' exists such that $F(z'|\beta)$, while $F(z'|r_vy)$ is true, whatever resultant r_v of β we choose to consider, then, since z itself is a resultant of β, $F(z'|zy)$, while, by the definition of z, $F(z|z'y)$, and hence $z = z'$. Therefore z may be taken as the unique representative of its own class of equivalents. Meanwhile, x possesses, with reference to β and y, the property of being a resultant of β such that every element of β is a mediator between x and y. Consequently, since $F(b_n|xy)$, $F(b_n\bar{y}|x)$, and hence x is a mediator between $\bar{y}$ and whatever element b_n of β may have been chosen. By 91, x is accordingly a mediator between $\bar{y}$ and whatever resultant, r_v, of β, may have been chosen. Whatever resultant, x', of β, possesses the property just ascribed to x, is such that $x = x'$.

101. *Conjugate limits of a collection with reference to a base.* We may sum up the result of the foregoing thus: If any element y be chosen, at our pleasure, as what we shall now call a base, and if hereupon any collection β be considered with reference to this base, then there exists a pair, and (barring for the moment the consideration of equivalent elements), a *single* pair, of conjugate resultants of β, which is so related to y and to β that (1) one of these two resultants (which we shall now symbolize by p) is such that $F(p|b_ny)$ for every element of β, and $F(p|r_vy)$ for every resultant of β; while (2) the other of these resultants, which we shall now symbolize by s, is such that $F(b_n|ys)$, and $F(r_v|ys)$, i. e., such that $F(b_n\bar{y}|s)$, and $F(r_v\bar{y}|s)$. If γ be the collection of the totality of those elements of Σ which are mediators between y on the one hand and each and every element and resultant of β, separately considered, on the other hand (see 92), then p has the property of belonging at once to the collection γ, and to the collection ρ; where ρ is, as before, the collection of all the resultants of β. Any element possessing the property of belonging at once to γ and to ρ, is equivalent to p, which may therefore be viewed, for present purposes, as the unique representative of its own class of equivalent elements. Barring equivalent elements, then, the collections γ and ρ have *only* this element p in common. If γ' be the collection of

those elements of Σ' which are mediators between $\bar{y}$ on the one hand, and each and every resultant of β, separately considered, on the other hand, then s has the property of belonging at once to the collection γ', and to the collection ρ. Any element possessing the property of belonging at once to γ' and to ρ, is equivalent to s, which may therefore be viewed, for the purposes of forming O-collections and F-collections, as the unique representative of its own class of equivalent elements. Barring equivalent elements, then, the collections γ' and ρ have *only* the element s in common. Of the pair (p, s), each resultant of β is a mediator. If, for the collection β, and for ρ, the collection of the totality of the resultants of β, the pair (p, s) alone is substituted, and if hereupon this pair is treated precisely as, in the foregoing, β itself has been treated, that is, if the resultants of (p, s) are first defined, and then their collection, viz., ρ, is compared with y, the same pair (p, s), is once more found as that pair of conjugate resultants of the collection (p, s) itself, whose relation to y is the relation heretofore characterized.

We shall now call the pair (p, s) a *pair of conjugate limits of β with reference to the base y*. For a given collection β, and for a given base y, principle VI thus requires us to define one such pair, and (barring equivalent elements), but a single pair, viz., (p, s), which may be viewed as *the* pair of conjugate limits in question. This pair, being a pair of mutually conjugate resultants of β, is symmetrically disposed with reference to the collection of the resultants of β. But, as has appeared in the foregoing, the pair (p, s) is not, in general, symmetrically disposed with respect to the enlarged collection (β, y). For p is a mediator between y and each resultant of β separately considered; while each resultant of β, separately considered, is a mediator of the pair (y, s). To mark this difference of relative position of p and s we may call: p the inferior limit of β with respect to y; s the superior limit of β with respect to y; while y is the base of this pair of conjugate limits of β. It is at once obvious that if we choose $\bar{y}$ as base instead of y, s would become the inferior, and p the superior limit of β with respect to $\bar{y}$.

102. We are now in a position to extend our result from the case of a single collection β, to the case of a set of collections β, γ, δ, etc., and in fact to the set of all possible collections of the elements of Σ. Holding a given base, y, chosen at pleasure, constant, we may consider any and all collections of the elements of Σ with reference to this one chosen base. If we do so, then, whatever collection, α, β, γ, δ, etc., we select, we shall find, by a process wholly analogous to the foregoing, that there exists an unique pair of resultants of any one such collection, such that this pair is, for that collection, the pair of conjugate limits of the collection with respect to y. Of this pair, one element is the superior, and the one the inferior limit of the collection in question, with respect to this chosen and constant base, while the choice of the base is arbitrary. The

limits are each time functions of the collection in question, and may be regarded as functions of that alone, *so long as the base is held constant.* *

If, while the base itself remains constant, a collection is altered, by the adjunction, or by the omission of elements, its limits alter or remain invariant, in ways whose laws are now to be defined. If various collections are considered with reference to the same base, their respective superior or inferior limits may be considered as new collections, with results whose laws are also to be determined. But if, instead, while any collection or sets of collections remain constant, the base is altered, so that an element y' or y'' takes the place of y, then the pairs of conjugate limits of any given collection with reference to the new base remain invariant, or alter, in accordance with still other principles (which we shall consider in chapter VI). We shall discuss these various cases in order. But in the rest of the present chapter, the base shall remain constant.

103. *Adjunction of the base to a collection.* If, to the collection β, while the base remains constant, the base y is itself adjoined, the inferior limit of (β, y) becomes equivalent to y itself; but the superior limit remains invariant. This appears from the reasoning used in 100.

104. *Adjunction of the obverse of the base; adjunction of resultants; other cases.* If the base y is held constant, and if $\bar{y}$ is thereupon adjoined to β, the inferior limit remains constant, while the superior limit of $(\beta, \bar{y})$ becomes equivalent to $\bar{y}$. The reasoning used in 100 can be employed to prove this also.—If any resultant of β is adjoined to β, or if any collection of the resultants of β is adjoined to β, the superior and inferior limits of the enlarged collection, so long as the base y is held constant, remain invariant; as appears from the reasoning used in 91, 92. Hence, by 89, if to a given collection, any mediator of any pair of its elements or of its resultants is adjoined, the limits remain constant; as they also do if the elements of a collection are repeated any multitude of times. If y is the constant base, and if any element or collection of elements of the collection γ, of 92 and 101, be adjoined to β, then, while the inferior limit, in general, alters, the superior limit of the collection remains in so far invariant. If any element or collection of elements chosen from the collection γ' of 101 be adjoined to β, this adjunction leaves the inferior limit of the enlarged collection invariant, while altering, in general, the superior limit.

Elements may be omitted from collections in a manner which is subject to these same laws. Thus, the omission of such repetitions of elements as occur in a collection does not alter either the superior or the inferior limits, etc.

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* The conception that the elements usually known as the products and sums of the algebra of logic are relative to a chosen constant base (the zero of the usual algebra of logic), and that the choice of what element of Σ shall be treated as the zero-element is essentially arbitrary, is Kempe's. But Kempe develops this conception solely on the basis of his theory of the symmetrical and unsymmetrical resultants of *triads*. By the more general concept of the conjugate resultants of collections, I have generalized Kempe's theory so as to be able to apply it, from the start, to collections of any multitude whatever.

The operation of determining the superior and inferior limits of a collection is obviously independent of the order in which the elements are arranged or considered, and may consequently be called a *commutative operation.*

105. *The transformation of a collection into its obverse collection.* If a collection β be transformed into its obverse collection (the base y remaining constant), if p be the inferior, and s be the superior limit of β, and if p' be the inferior and s' be the superior limit of $\bar{\beta}$ then the two equivalences hold good:

$$\bar{p} = s'\,; \text{ and } \bar{s} = p'.$$

For, by the definition of an inferior limit, $F(p\,|\,yr_v)$ for every resultant, r_v, of β, while $F(p\,|\,\beta)$. Whence there follows (1) $F(\bar{p}\,|\,\bar{y}\bar{r}_v)$ for every resultant $\bar{r}_v$ of $\bar{\beta}$; while (2) $F(\bar{p}\,|\,\bar{\beta})$. From (2) it follows that $\bar{p}$ is a determinate resultant of $\bar{\beta}$. From (1) follows, by transfer, $F(\bar{p}y\,|\,\bar{r}_v)$, whatever resultant $\bar{r}_v$ of $\bar{\beta}$ may be chosen. Thus $\bar{p}$ is such a resultant of $\bar{\beta}$ that whatever element or resultant $\bar{r}_v$ of $\bar{\beta}$ may be chosen, this element or resultant of $\bar{\beta}$ is a mediator between $\bar{p}$ and y. Hence $\bar{p}$ is equivalent to the superior limit of $\bar{\beta}$ with respect to the base y. Hence $s' = \bar{p}$.

Furthermore, since s is the superior limit of β with respect to y, we have $F(r_v\,|\,sy)$, whatever resultant, r_v, of β we may choose; while, at the same time $F(s\,|\,\beta)$. It follows that $F(\bar{s}\,|\,\bar{\beta})$, so that $\bar{s}$ is a determinate resultant of $\bar{\beta}$; while $F(\bar{r}_v\,|\,\bar{s}\bar{y})$, i. e., $F(\bar{r}_v y\,|\,\bar{s})$, for every resultant $\bar{r}_v$ of $\bar{\beta}$; so that $\bar{s}$ is a mediator between y on the one hand, and each and every $\bar{r}_v$ of $\bar{\beta}$ on the other hand. Thus $\bar{s}$ is equivalent to the inferior limit of $\bar{\beta}$ for the base y. The equivalences in question, viz., $\bar{p} = s'$; and $\bar{s} = p'$, are accordingly proved.

106. *The combination of collections. The associative law in the determination of pairs of conjugate limits with respect to a constant base.* If we consider the inferior limit of a collection β_1, with respect to a base y, and also the inferior limits, with respect to the same base, of collections $\beta_2, \beta_3, \cdots, \beta_n, \cdots$, in any multitude of collections, and if π be the collection of all these inferior limits, while ω is the collection consisting of the totality of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, then the inferior limit of π is equivalent to the inferior limit of ω (the subscripts $1, 2, \cdots, n, \cdots$ are now no longer to be viewed as ordinal numbers, but merely as distinguishing marks).

The inferior limit p_ω of the collection ω is, in fact, an element such that $F(\omega\,|\,p_\omega)$, while if $b_n^{(k)}$ is any element of any collection β_n in the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, $F(p_\omega\,|\,yb_n^{(k)})$. Meanwhile, if p_n is the inferior limit of any collection β_n, then $F(\beta_n\,|\,p_n)$, while $F(p_n\,|\,b_n^{(k)}y)$ for every element $b_n^{(k)}$ of β_n. If π is the collection $(p_1, p_2, \cdots, p_n, \cdots)$, then its inferior limit p_π is an element such that $F(\pi\,|\,p_\pi)$, while, whatever p_n may be selected, $F(p_\pi\,|\,p_n y)$. Since $F(p_n\,|\,b_n^{(k)}y)$, and $F(p_\pi\,|\,p_n y)$, it follows by 71, that $F(p_\pi\,|\,b_n^{(k)}y)$; and this latter assertion holds true, whatever element $b_n^{(k)}$ of any collection β_n

may be selected, that is, whatever element of ω may be selected. Moreover, whatever element p_n of π may be selected, since p_n is a resultant of β_n, p_n is also a resultant of ω (88). Thus $F(\omega|p_n)$, whatever element p_n of π may be selected, while $F(\pi|p_\pi)$. Hence (by 57), $F(\omega|p_\pi)$. Thus p_π is a resultant of ω, while p_π is such that $F(p_\pi|b_n^{(k)}y)$ for every element $b_n^{(k)}$ of ω. Hence $p_\pi = p_\omega$.

The operation of determining the inferior limit of a collection ω which is composed of a set of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, is consequently *associative* with respect to the operation of separately determining the inferior limit of each one of these collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$.

By a precisely similar reasoning, one can obviously prove that the operation of determining the superior limit, s_ω, of the collection ω, is associative, in case we separately determine the superior limits of β_1, β_2, etc., and use such determination as the basis for determining s_ω.

107. *Cross collections.* If a set of collections $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$ is given (where the multitude of collections in question is wholly unrestricted), and if a collection λ is formed by selecting, at pleasure, one element, and one only (say $b_1^{(i)}$), from β_1, one element, and one only (say $b_2^{(j)}$), from β_2, and one element, and only one, from each of the collections of the set (so that, for instance, $b_n^{(v)}$ is selected from β_n), then the collection λ_z, that is, the collection $(b_1^{(i)}, b_2^{(j)}, \cdots, b_n^{(v)}, \cdots)$ shall be called a *cross-collection* of the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$. As many distinct cross-collections $(\lambda_t, \cdots, \lambda_x, \cdots, \lambda_z, \cdots)$ exist as there are distinct possible combinations of elements selected one from each of the collections of the set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$. Choose now a constant base y, to be retained throughout what follows in this and in the next section. Hereupon, let s_n be the superior limit with respect to y of the collection β_n of the original set. Let $(s_1, s_2, \cdots, s_n, \cdots)$ be the collection of all such superior limits, with respect to y, of β_1, β_2, etc. Let the collection $(s_1, s_2, \cdots, s_n, \cdots)$ be here called the collection ϑ. Let the inferior limit of ϑ, with respect to the base y, be symbolized by p_ϑ. Next, let p_z be the inferior limit of λ_z, that is, of the collection $(b_1^{(i)}, b_2^{(j)}, \cdots, b_n^{(v)}, \cdots)$. Let $(p_t, \cdots, p_x, \cdots, p_z, \cdots)$ be the collection comprising *all* such inferior limits of the possible cross-collections $(\lambda_t, \cdots, \lambda_x, \cdots, \lambda_z, \cdots)$. Let the collection $(p_t, \cdots, p_x, \cdots, p_z, \cdots)$ be called the collection ψ. Let the superior limit of ψ be symbolized by s_ψ. Then we shall next inquire how p_ϑ is related to s_ψ, that is, how the inferior limit of the collection of all the respective superior limits of a given set of collections, is related to the superior limit of the collection of all the respective inferior limits of the corresponding cross-collections. To ask this question is to inquire (in the most general form possible) whether the operation of seeking inferior limits is *distributive* with reference to the operation of seeking the superior limits of given collections.

108. *The Distributive Law.* Using the conventions of the previous section,

regarding the cross-collections, and retaining the symbols used, we are able to assert that

$$p_\vartheta = s_\psi.$$

For, by the definition of an inferior limit, whatever element p_z we may select from the collection ψ, the element p_z is such that $F(p_z | y b_n^{(v)})$ for every element $b_n^{(v)}$ which belongs to the cross-collection λ_z. And, by the rule according to which this collection λ_z has been formed, there exists, in each of the original collections β_n, an element, $b_n^{(v)}$, for which the foregoing assertion is true. Meanwhile, if we consider the relation of the element $b_n^{(v)}$ to its own collection β_n, of the original set $(\beta_1, \beta_2, \cdots, \beta_n, \cdots)$, it follows, by the definition of a superior limit, that $F(b_n^{(v)} | s_n y)$. Since, then, the element $b_n^{(v)}$ is such that $F(p_z | y b_n^{(v)})$, that is, such that $F(b_n^{(v)} | p_z \bar{y})$, while $F(b_n^{(v)} | s_n y)$, we have, by the usual rule for the elimination of an element, $F(s_n y | p_z \bar{y})$; whence follows $F(p_z | s_n y)$. Since, whatever p_z may be in question, that element of λ_z which λ_z has in common with any given collection β_n may thus be eliminated, it follows that the relation $F(p_z | s_n y)$ holds of every element p_z of the collection ψ, when the relation of p_z to any element, whatever s_n, of ϑ, is considered. Since, however, by the definition of a superior limit, $F(s_\psi | \psi)$ is true, while $F(p_z | s_n y)$ holds true of each element, p_z, of ψ, separately considered, in its relation to each element of ϑ, separately considered, it follows, by 57, that, whatever s_n be chosen, the pair (s_n, y) is such that (1) $F(s_n y | s_\psi)$.

Furthermore, since, by the definition of a superior limit, whatever p_z we may select $F(p_z | s_\psi y)$, while, by the definition of an inferior limit, $F(p_z | \lambda_z)$, it follows that, whatever λ_z we may select, $F(\lambda_z | s_\psi y)$ is always true. Since this latter relation is general, and holds for every collection λ_z, without exception, we have, for every possible cross-collection, $F(b_1^{(i)} b_2^{(j)} \cdots b_n^{(v)} \cdots | s_\psi y)$; an assertion which remains invariant and true if, instead of selecting, from any one of the collections $(\beta_1, \beta_2 \cdots \beta_n \cdots)$ the element which here appears in the collection λ_z, we select instead any other of the elements of that same collection β_1, β_2, etc. Consider, hereupon, once more, the element $b_n^{(v)}$, here selected from the collection β_n. Let $\lambda_z^{(n)}$ be what the collection λ_z becomes if that element in λ_z which is derived from the collection β_n is *omitted*. In other words, let λ_z be so subjected to partition that λ_z is the same collection as $(\lambda_z^{(n)}, b_n^{(v)})$. Then we have, from the foregoing, $F(\lambda_z^{(n)} b_n^{(v)} | s_\psi y)$, that is, by transfer, $F(b_n^{(v)} | s_\psi y \bar{\lambda}_z^{(n)})$; while this relation holds true whatever element $b_n^{(v)}$ is selected from β_n. Since, however, s_n is such that $F(s_n | \beta_n)$, while for $b_n^{(v)}$ any element whatever of β_n may be substituted, we have, by 57, the collection $(s_\psi, y, \bar{\lambda}_z^{(n)})$ such that $F(s_n | s_\psi y \bar{\lambda}_z^{(n)})$; so that, by transfer, $F(\lambda_z^{(n)} s_n | s_\psi y)$. It follows that, while, as before, $F(\lambda_z | s_\psi y)$ is true, this relation remains invariant if *we substitute for any element* $b_n^{(v)}$, *of* λ_z, *the corresponding superior limit*, s_n, *of that collection* β_n, *from which* $b_n^{(v)}$ *has* *been selected.* Such substitution may be accomplished in case of any element $b_1^{(i)}$, $b_2^{(j)}$, etc., independently of whether such a substitution has been made in case of any other element of λ_z.

For since both the assertions $F(\lambda_z | s_\psi y)$, and $F(\lambda_z^{(m)} s_m | s_\psi y)$, hold good in case an element $b_m^{(q)}$ of λ_z is selected from any collection β_m which is *not* the collection β_n, we could, by a repetition of the foregoing process of reasoning, show that the relation expressed in $F(\lambda_z | s_\psi y)$ not only remains invariant whichever one of the two elements, $b_n^{(v)}$ or $b_m^{(q)}$ is selected as that element for which the corresponding s_n or s_m is substituted; but *also* remains invariant when for both of them, simultaneously, the corresponding superior limits of β_n and β_m are substituted. This result can be extended, at pleasure, therefore, to any number, or to any partial collection, or to *all* of the elements of λ_z, without regard to their multitude. The relation $F(\lambda_z | s_\psi y)$ therefore remains invariant in case, for *each and every* member of λ_z, we substitute the corresponding element s_n, viz., the superior limit of that collection β_n from which the member of λ which is each time in question was itself selected. Hence carrying out this substitution, we have (2) $F(\vartheta | s_\psi y)$.

But we have seen above, by (1), that $F(s_n y | s_\psi)$ is true. By adjunction it follows from this that $F(\vartheta y | s_\psi)$ is true. Hence we have at once true the *two* assertions (2) $F(y | \vartheta \bar{s}_\psi)$ and $F(y | \bar{\vartheta} s_\psi)$. Hence $F(\vartheta \bar{s}_\psi | \bar{\vartheta} s_\psi)$. Hence (3) $F(\vartheta | s_\psi)$, by 54.

By (1), therefore, $F(s_n y | s_\psi)$ is true, whatever element s_n of ϑ we may choose; while, by (3), $F(s_\psi | \vartheta)$, so that s_ψ is a resultant of ϑ. It follows that s_ψ is an element such that it is a mediator between y and every element of ϑ separately considered while s_ψ is also a resultant of ϑ. Hence, by the definition of an inferior limit, $s_\psi = p_\vartheta$; and the theorem is proved.*

109. *The second form of the distributive law.* Still retaining constant the base y, let s_z be the superior limit of any cross-collection λ_z, as such collections were defined in 107. Let the collection of all the superior limits of the cross-

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* The demonstration of the distributive law here given may be regarded as a generalization of KEMPE's treatment of the symmetrical resultants of triads; although this generalization involves considerations which are somewhat peculiar to the present form of the theory of conjugate resultants. When the relations of the F-collections are regarded as degenerating into the specialized but more familiar relation of antecedent and consequent, the proof of the distributive law becomes subject to those difficulties whose treatment by Mr. C. S. PEIRCE, by SCHROEDER, and by Dr. HUNTINGTON, are summed up by Dr. HUNTINGTON in his *Sets of Postulates for the Algebra of Logic* (these Transactions, July, 1904). The difficulties in question are a test of the sort and of the amount of information which is surrendered when, instead of the F-relations viewed, so to speak, in their entirety, we confine ourselves to relations which are defined, for all the collections concerned, merely with reference to a common origin, and when we thereupon define the usual logical "sums" and "products" solely upon the basis of the antecedent-consequent relations. KEMPE's Theory, like the usual one, extends the distributive law by induction from pairs of triads to any number of cases. The present treatment, in this paper, applies the distributive law at once to collections of any multitude whatever.

collections $(s_t, \cdots, s_x, \cdots, s_z, \cdots)$, be called the collection ϕ. Let the inferior limit of ϕ be p_ϕ. Let the inferior limit of any collection β_n be p_n. Let the collection $(p_1, p_2, \cdots, p_n, \cdots)$ of all such inferior limits be the collection ϵ. Let the superior limit of ϵ be s_ϵ. Then

$$p_\phi = s_\epsilon.$$

For whatever element $b_n^{(v)}$ we may select from the cross-collection λ_z, $F(s_z y \mid b_n^{(v)})$ is true. And whatever collection β_n we may select, and whatever λ_z is in question, there exists in β_n an element for which the assertion $F(s_z y \mid b_n^{(v)})$ is true. But, in case of the collection β_n, the assertion $F(y b_n^{(v)} \mid p_n)$ is true. Hence, since $F(b_n^{(v)} \mid p_n \bar{y})$ and $F(b_n^{(v)} \mid s_z y)$, it follows that $F(s_z y \mid p_n)$; an assertion which is true of every p_n in its relation to every s_z. From $F(s_z y \mid p_n)$, follows $F(s_z \mid \bar{y} p_n)$, an assertion which is true of every element s_z of the collection ϕ, while $F(\phi \mid p_\phi)$. Hence the collection $(\bar{y}, p_n)$ is such that $F(\bar{y} p_n \mid p_\phi)$, or (1) $F(p_n \mid p_\phi y)$; an assertion which again holds true whatever p_n of ϵ may be selected.

Furthermore, since $F(p_\phi \mid y s_z)$, whatever s_z may be selected, while, by the definition of s_z, $F(s_z \mid \lambda_z)$, we obtain, by the usual elimination process, $F(p_\phi \bar{y} \mid \lambda_z)$ or $F(p_\phi \bar{y} \mid b_1^{(i)} b_2^{(j)} \cdots b_n^{(v)} \cdots)$. This relation remains invariant whatever element of β_1 be substituted for $b_1^{(i)}$; whatever element of β_2 be substituted for $b_2^{(j)}$; whatever element of β_n be substituted for $b_n^{(v)}$; and so on. If, as in 108, $\lambda_z^{(n)}$ be what λ_z becomes when $b_n^{(v)}$ is omitted, we have, by transfer of the entire collection $\lambda_z^{(n)}$, $F(p_\phi y \bar{\lambda}_z^{(n)} \mid b_n^{(v)})$; and this holds true for every element $b_n^{(v)}$ of β_n, separately considered. But $F(p_n \mid \beta_n)$. Hence the collection $(p_\phi, \bar{y}, \bar{\lambda}_z^{(n)})$ forms an F-collection with every element of β_n, separately considered, while $F(p_n \mid \beta_n)$; and hence $F(p_\phi \bar{y} \bar{\lambda}_z^{(n)} \mid p_n)$; that is, $F(p_\phi \bar{y} \mid \lambda_z^{(n)} p_n)$. And thus for any element $b_1^{(i)}$, $b_2^{(j)}$, $b_m^{(q)}$, $b_n^{(v)}$, of λ_z, there may be substituted, either simultaneously with or independently of, any of the other elements, the inferior limit, p_1, p_2, p_m, p_n of the collection β_1, β_2, β_m, β_n, from which the element in question is selected. If the substitution is effected simultaneously for all the elements of λ_z, it follows that $F(p_\phi \bar{y} \mid p_1 p_2 \cdots p_n \cdots)$; that is, by transfer (2) $F(p_\phi \bar{\epsilon} \mid y)$.

But by (1) $F(p_n \mid p_\phi y)$. By the adjunction of all the elements of ϵ besides p_n this becomes $F(\epsilon \mid p_\phi y)$; that is $F(\epsilon \bar{p}_\phi \mid y)$. By the elimination of y it follows, from (1) and (2), that $F(\epsilon \bar{p}_\phi \mid p_\phi \bar{\epsilon})$. Hence $F(\epsilon \mid p_\phi)$. The element p_ϕ is so related to ϵ that p_ϕ is a resultant of ϵ, while, by (1), whatever element p_n of ϵ be selected, $F(p_n \mid p_\phi y)$. Hence p_ϕ is equivalent to the superior limit of ϵ, and:

$$p_\phi = s_\epsilon,$$

as was to be proved.

Thus the operation of seeking the superior limits is distributive with reference to the operation of seeking the inferior limits of given collections.

110. The results of 91–109 may now be summarized in the somewhat less general, but (by virtue of some of our best established mental habits) more easily apprehended form of a series of statements concerning the antecedents and consequents of the members of one or more collections.

Let an element y be chosen, and held constant, *both as the base and as the origin with reference to which antecedents and consequents* are to be determined. What we have shown is:

111. That if a given collection β be considered, then, whatever element c is an antecedent, with respect to y, of every element of β, separately considered, is also an antecedent, with respect to y, of every resultant of β (see 91). Moreover (by 101; see what is there said concerning the collection γ'), whatever element c' is a consequent of every element of β, is also a consequent of every resultant of β. If we consider: (I) the collection γ, consisting of all elements c, each of which is an antecedent of every resultant of β; (II) the collection ρ, consisting of all the resultants and of all the elements of β; and (III) the collection γ', consisting of all the elements c', each of which is a consequent of every element of β: then each of these three collections is a collection which includes all of its own resultants. Each therefore is an internally complete or "perfect" collection.

112. If β includes y, or an element equivalent to y, then the collection γ reduces to the single element y itself. If β includes $\bar{y}$, or any element equivalent to $\bar{y}$, then the collection γ' reduces to $\bar{y}$. If β is an O-collection, the same result obtains for both γ and $\bar{\gamma}$; that is γ reduces to y and γ' to $\bar{y}$ (95), all the equivalents of an element being here regarded as represented by that element.

113. The collections γ and ρ have an element in common (101). This element is what we have called p, the inferior limit of β with respect to y. Whatever element γ and ρ have in common is equivalent to p. The collections ρ and γ' have an element in common, viz., s, the superior limit of β with respect to y. Whatever element γ' and ρ have in common is equivalent to s. If all the elements of β are mutually equivalent, for instance, if they are all equivalent to b, then $b = p = s$. Otherwise, $p \neq s$. The elements p and s remain invariant in case elements of β are repeated, any multitude of times, or in case any resultants of β are adjoined to β. Moreover p remains invariant whatever elements of γ' are adjoined to β; and s remains invariant, whatever elements of γ are adjoined to β (103, 104).

114. The element p is definable as an element which is: (I) An antecedent of every element (and so of every resultant) of β; and (II) an element such that, whatever element of Σ is an antecedent of every element of β, is also an antecedent of p. Or again, p may be defined as that antecedent, of every resultant of β, which is also itself a resultant of β. Or finally, p may be defined as that antecedent of every element of β which is itself a consequent of every element c which agrees with p in being an antecedent of every element of β.

115. The element s is definable as an element which is: (I) A consequent of every element (and so of every resultant) of β; and (II) an element such that, whatever element of Σ is a consequent of every element of β, is also a consequent of s. Or again, s may be defined as that consequent, of every resultant of β, which is also itself a resultant of β. Or, finally, s may be defined as that consequent of every element of β which is itself an antecedent of every element c' which agrees with s in being a consequent of every element of β.

116. If β is an E-collection, no one of whose elements is equivalent to y, and no one of whose elements is equivalent to $\bar{y}$, while β itself contains at least one pair of non-equivalent elements, then, by 89, and by the reasoning of chapter V, the three collections, γ, ρ and γ', contain, each of them, an infinite number of elements.

117. Of the various collections here in question a principle holds true which is statable in general, on the basis of the foregoing, as a consequence of principle VI, and as holding throughout the system Σ: If two collections are such that one of them (say γ) includes all those elements of Σ each of which is an antecedent with respect to a given origin y, of every element of the other collection (say β), then there exists in Σ an element (the inferior limit, p, of β, with respect to y), which is at once a member of the collection γ, and also a resultant of β. Or again: If two collections α and β exist, such that every element of α is an antecedent, with respect to y, of every element of β, then there exists at least one element of Σ which is a consequent of every element of α, and which is also an antecedent of every element of β. The superior limit of α with respect to y, and the inferior limit of β with respect to y, both of them stand in this position. If they are not mutually equivalent, all the elements which are their mediators agree in possessing the character in question. Another and more restricted form of the same principle runs thus: Whatever infinite sequence κ, consisting of elements of Σ, is so definable that, with reference to a chosen origin, every element k_r of the sequence possesses a consequent k_t which also belongs to the sequence, there also exists in Σ an element which is a consequent of every element of κ. For κ is a collection of elements of Σ, and consequently possesses, by the foregoing, a superior limit, with respect to y, which also belongs to Σ. The chains of elements, defined in chapter V, consequently all of them possess superior limits belonging to Σ.*

118. It is now possible, without further difficulty, to point out that the elements of Σ possess the properties of a system of logical classes, or of entities to which the ordinary algebra of logic applies. Let the arbitrarily assumed origin

*The definite relation thus brought out between the conceptions of logical products and sums, and the conception of limits, is, so far as I know, a new feature of the present discussion. It is brought to light by defining, from the outset, these conceptions with reference to collections of unrestricted multitude.

y be taken as the 0 of the ordinary algebra of logic. Let $\bar{y}$ be taken as the 1 of that algebra (otherwise symbolized, by some, as ∞). Let the relation $p \prec_y q$ be regarded as the usual relation of logical antecedent and consequent; and let the subscript of the symbol $\prec_y$ be dropped. by virtue of that usual convention which regards the reference to 0, not as reference to an arbitrary origin, but as such that $a \prec b$ has an invariant or absolute sense. So regarded, the system Σ possesses an element, 0, such that whatever element x be chosen, $0 \prec x$; and also an element, 1, such that whatever element z be chosen, $z \prec 1$. The relation $\prec$ is transitive. If $a \prec b$ and $b \prec a$, then $a = b$. If $a \prec b$, and also $a \neq b$, the relation $\prec$ is asymmetrical. Elements such as p and s may first be viewed as determined by some given pair of elements, e. g., by the pair (a, b). The element p is then called the *product*, the element s is called the *sum* of this pair; and, in the usual symbols, one may write

$$ab = p; \qquad a + b = s.$$

The definitions of the operations of logical multiplication and of logical addition, may assume the form explained in 114 and 115. Obverses will now appear as elements each of which is what is ordinarily called the negative of the other. Since, in fact, by 95 and 112, the product of an O-collection is the origin, and its sum is the obverse of the origin (see also 80, 82), the obverse elements a and $\bar{a}$ are such that $a + \bar{a} = 1$, while $a\bar{a} = 0$. We shall have the known results (easily verifiable on the basis of the foregoing):

$$a + a = a; \qquad aa = a; \qquad a0 = 0; \qquad a + 0 = a;$$

$$a\ 1 = a; \qquad a + 1 = 1; \qquad \overline{a + b} = \bar{a}\bar{b}; \qquad \overline{ab} = \bar{a} + \bar{b};$$

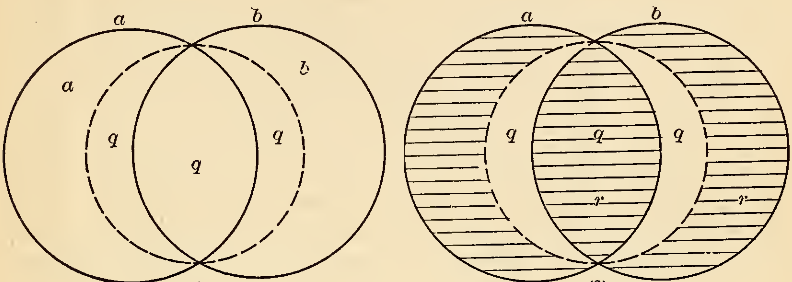
$$(ab)c = a(bc); \qquad a(b + c) = ab + ac; \qquad a + bc = (a + b)(a + c).$$

Not only are these results predetermined by the foregoing discussion, but we have in fact given to the principles in question a form much more general than the usual form by so stating the principles from the start that they apply to logical operations upon collections possessing any multitude whatever.

It follows then, that the usual algebra of logic applies without restriction to the system Σ, which is in so far identical with a totality of logical classes, whereof an infinity are mutually non-equivalent, while all are capable of an unrestricted combination by the operations of logical addition and of logical multiplication.

It is worthy of note that, in terms of the ordinary algebra, the conjugate resultants of a given collection may be defined as follows: Let the logical product of a collection β of logical elements be p. Let s be the sum of β. Then any element q such that $p \prec q \prec s$ is a resultant of β. If r is a resultant of β

such that $qr = p$, while $q + r = s$, then q and r are conjugate resultants of β. If β is the pair (a, b) represented in the diagram (1), and if q is a resultant of (a, b) so that q includes the common part of (a, b), but does not extend beyond the limits of $a + b$, then, by repeating (a, b) in the diagram (2), we may indicate, by shading, the portion of the repeated diagram where r lies, and



(1) (2)

so the extent of r, the conjugate resultant of q, in the pair (a, b). It will be observed that q and r have the product ab in common, but supplement each other as to the remainder of $a + b$. If q expands so as to coincide with the whole of $a + b$, r shrinks to ab, and conversely.

The negatives $\bar{q}$ and $\bar{r}$ are the elements whose existence is directly asserted in principle VI.

Chapter VI. The system Σ as a generalized space-form.

119. The inquiry of the previous chapter was primarily devoted to determining what elements exist in Σ, and how they are arranged. As an incident to this research, the relations of our system Σ to the system defined in the algebra of logic was developed. But the consequences of principles V and VI, in their combination, have still other aspects. In particular, the properties of the system Σ to which we have already called attention, make its array analogous to that of the points of space. This we are next to see.

120. By definition, all of the elements of any collection are resultants of that collection. But if a collection β contains at least one pair of non-equivalent elements, and if β at the same time comprises only a finite collection of the resultants of some one of those pairs of non-equivalent elements which β contains, then there exists an infinite collection of resultants of β such that these resultants are not themselves elements of β. This appears from 89, in combination with principle V. If β is an O-collection, all the elements of Σ are resultants of β, and are also complements of β. If β is not an O-collection, Σ contains an infinity of elements which are not resultants of β. For the

obverses of the resultants of β can none of them be resultants of β; since if $F(\beta|x)$ and $F(\beta|\bar{x})$ are both of them true, $O(\beta)$. The obverses of the resultants of β are, by definition, complements of β. But if β is not an O-collection, Σ also contains an infinity of elements which are *neither* complements nor resultants of β. For let (a, b) be any pair of conjugate resultants of β, so that $J(ab; \beta)$. Since β is not an O-collection, $E(ab)$ is true (by 95). Every resultant of β is a mediator of the pair (a, b). Since $E(ab)$, $a \mp \bar{b}$ and $b \mp \bar{a}$. No element x such that $F(x|\bar{a}b)$ is true, and no element x', such that $F(x'|a\bar{b})$ is true, can be a resultant of β unless $x = b$ or $x' = a$. Since the complements of β, being obverses of the resultants of β, are mediators of the pair $(\bar{a}, \bar{b})$ we can thus define an infinity of elements which are neither complements of β nor resultants of β. All elements, x, such that $x \mp \bar{a}$, $x \mp b$ and $F(x|\bar{a}b)$; and all elements x', such that $x' \mp a$, $x' \mp \bar{b}$ and $F(x'|a\bar{b})$, are, namely, neither complements nor resultants of β. An analogous assertion holds for any other of the pairs of conjugate resultants of the col lection β. Whatever pair of elements (a, b) we may choose, an infinity of pairs of elements of the form $(a, \bar{b})$, $(b, \bar{a})(c, \bar{d})$, $(d, \bar{c})(e, \bar{f})$, $(f, \bar{e})$, etc., are thereby determined, such that each pair consists of some resultant c, d, e of the pair (a, b), while the other member of each pair is the obverse of the corresponding conjugate resultant of (a, b), No mediator of any one of the pairs $(a, \bar{b})$, $(c, \bar{d})$, $(\bar{e}, f)$, etc. (except a, or c, or f, as the case may be), is a resultant of the pair (a, b).

If a, b, c, d, e, f, are mutually non-equivalent elements, while the pairs (a, b), (c, d), (e, f), are each of them pairs of conjugate resultants of the same pair, or of the same collection β, then no resultant of $(\bar{a}, b)$, or of $(a, \bar{b})$, or of $(c, \bar{d})$, or of $(\bar{c}, d)$, or of $(e, \bar{f})$, can be equivalent to any resultant of the other pairs thus defined. If x, for instance, is a mediator of $(c, \bar{d})$, so that $F(x|c\bar{d})$, $F(dx|c)$, then, with respect to x, taken as a base, the conjugate limits of (a, b), or of the collection β, in question, are, respectively c and d; c being the inferior and d the superior limit with respect to the base x, of the collection β. If therefore x' is such that $F(x'|e\bar{f})$, x' can be equivalent to x only in case each of these elements x and x', taken as base, determines, in the manner shown in 100, 101, the same pair of conjugate limits of β; in which case $c = e$, $d = f$. For if $F(dx|c)$, and $F(xf|e)$, where (c, d), and (e, f) are conjugate pairs, this result follows.

121. If we begin afresh, with a pair (a, b), and then choose a base, y, such that $F(y|a\bar{b})$, it is thus plain that, for this base y, a is the inferior, and b is the superior limit of the collection of the resultants of (a, b), and that this choice of inferior and superior limits for (a, b), *remains invariant for any base that is a mediator* of $(a, \bar{b})$, while, if the base is changed to some other pair $(c, \bar{d})$, $(d, \bar{c})$, etc. — (c, d) being a pair of conjugate resultants — the inferior

and superior limits of (a, b) (the products and sums of the ordinary algebra of logic) vary accordingly. The totality of the expressions employed in the ordinary algebra of logic to represent the relations of a system of classes, will remain invariant as to certain values, and undergoes, for other values, perfectly definite transformations, in case the base with reference to which products and sums are reckoned is altered, so that some class y takes the place which has been assigned to the zero of the ordinary algebra. If the base is, for instance, changed from some element y', such that, $F(y' | a\bar{b})$, to some element y, such that, $F(y | \bar{a}b)$, the product of the pair (a, b) is transformed into what was formerly its sum, and the sum into what was formerly the product. If the new base is an element y such that $F(y | \bar{a}\bar{b})$, the product and sum of (a, b) are transformed into a new pair of the conjugate resultants of (a, b). These transformations, somewhat analogous, for the algebra of logic, to a transformation of coördinates in a space system, lead to results which are predetermined by the F-relations of the elements of the system Σ. However the base may be changed, the product of any O-collection will be equivalent to the new base; the sum of an O-collection will be equivalent to the obverse of the base; and so on.

Meanwhile, any pair (a, b), such that $E(ab)$, while $a \neq b$, forms a means of an exhaustive classification of the elements of Σ. Given, namely, any element of Σ, say x, there is some determinate pair of resultants of (a, b), say the pair (k, l), such $F(x | k\bar{l})$. To the resultants of the pair $(k, \bar{l})$ belongs one resultant, k of (a, b), and there are also an infinite number of possible bases, for which k is product and l sum, of (a, b). No element of Σ belongs at once to two of the distinct classes thus defined by selecting pairs of conjugate resultants (i, j), (k, l), such that $i \neq k$, $i \neq l$, etc., and by then defining the class of the resultants of $(k, \bar{l})$ and of $(\bar{k}, l)$.

Or, again, one may express our present result by saying that if an element x is *not* a resultant of the pair (a, b), then there exists one and only one pair of conjugate resultants of (a, b), namely the pair (k, l), such that, if the obverse of a determinate one, say k, of these two conjugate resultants of (a, b) be chosen, $F(x | \bar{k}, l)$, while $x \neq l$.

122. If a pair of elements (a, b) be chosen such that $a \neq b$, and $E(ab)$, it is always possible to find a pair of resultants, (q, r) of (a, b), such that q and r are not mutually conjugate resultants of the pair (a, b), while $q \neq r$, and $E(qr)$, and while $F(a | qr)$ and $F(b | qr)$ are both of them false. In order to construct such a pair, it is only necessary to choose any resultant q, of the pair (a, b), such that $q \neq a$, $q \neq b$, and then to determine r such that $r \neq q$, $r \neq b$, and $F(r | qb)$. In this case, since $F(q | ab)$, and $F(r | qb)$, it is impossible that $F(a | qr)$. For if $F(a | qr)$ and $F(r | qb)$, it follows that $F(a | qb)$; while since, at the same time $F(q | ab)$, there results $q = a$, contrary to the hypothesis. Moreover, if $F(b | qr)$, and $F(r | qb)$, it follows that

$r = b$; and this again is counter to the hypothesis. Hence q and r are both of them resultants of (a, b), but neither a nor b is a resultant of (q, r).

Consider, now, the collection ρ of all the resultants of the pair (a, b), and the collection ρ_1 of the resultants of the pair (q, r). Every element of ρ_1 belongs to ρ. But there exists an infinity of elements of ρ such that no one of them either belongs to ρ_1 or is equivalent to any of the elements of ρ_1. Since the pair (q, r) is again a pair of non-equivalent elements which is not an O-pair, it is possible to determine new pairs (s, t), (u, v), (w, x), etc., without limit, such that the resultants of these pairs form a series, or chain, of collections, ρ, ρ_1, ρ_2, whereof each collection is wholly inclusive of all the elements of each later collection, while each collection contains an infinity of elements that are not included in the later collections, and that are equivalent to none of the elements so included.

It is, in the reverse direction, possible to include any collection ρ of the resultants of a given pair (so long as this is not an O-pair), in some more inclusive collection, ρ', which then may be enlarged, if necessary, to the collection of the resultants of some new pair, by considering any of the pairs of conjugate resultants of ρ'.

123. The structure of Σ is, therefore, such as to permit this endless determining of internally complete systems of resultants within systems, every such collection comprising an infinite set of elements. This being the case, the question arises whether there is also any sense in which the system Σ may be said to possess a "dimensionality" resembling that of space. The answer is that such a conception, in the system Σ, is capable of arbitrary definition in an infinite number of ways. And such a way, in fact, is suggested by the relation of any inclusive system ρ of the resultants of a pair (a, b), and any included system such as the collection ρ_1 of resultants of the pair (q, r) defined above.

Suppose, namely, that we arbitrarily define the collection of the resultants of the pair (q, r) as a one-dimensional collection, simply because the totality of these resultants is determined by the naming of the single pair of elements (q, r). In precisely the same sense, it would appear that the resultants of (a, b) or any other pair might be regarded as also of one dimension. But if we consider more carefully, it is plain that the following reason appears for a distinction between the systems ρ and ρ_1. Let m be any resultant of (q, r), such that $F(qr|m)$ while $m \neq q$, $m \neq r$. In ρ, that is, with respect to (a, b), m possesses a conjugate resultant n, such that $J[mn; (a, b)]$, i. e., $J(mn; \rho)$. Now it is plain that $F(qr|n)$ is false. For if $F(qr|n)$ and $F(qr|m)$ were both at once true, we should have, by 73, every element of ρ a mediator of (q, r), and so (q, r) would be a pair of conjugate resultants of (a, b), which is contrary to the construction as stated in 122. Consider the triad (q, r, n). Since $F(qr|m)$, any resultant of the pair (m, n) is a resultant of (q, r, n), as can readily be shown by the usual elimination-process. Hence any element of ρ is a resultant of the triad (q, r, n). Hence the triad (q, r, n) possesses resultants which are not resultants of any of the pairs (q, r), (r, n), (q, n). The triad (q, r, n) resembles then a triangle, or two-dimensional complex, when viewed with reference to the pairs (q, r), (r, n), (q, n) Thus ρ can be viewed as a two-dimensional complex in relation to ρ_1. An analogous result holds whatever pair (a, q), (b, r), etc., we choose from the resultants of ρ, so long as the resultants of this selected pair form only a portion of the resultants of ρ, while elements equivalent to none of the resultants of the selected pair belong to ρ.

But we are not limited in our selection to the whole system ρ, in order to be able to define such triads. Consider next the triad (q, b, n). By 122, q and r have been so defined that $F(r|qb)$. Since, by construction, $F(m|qr)$, it follows that $F(m|qb)$. Were n also such that $F(n|qb)$, every resultant of (a, b) would be also a resultant of (q, b), which is false by construction. Hence $F(n|qb)$ is false. Were $F(q|bn)$ true, then since, as just shown, $F(m|qb)$, we should have true $F(m|bn)$, and hence, since $F(b|mn)$, it would follow that $b = m$, which is impossible by construction. For $F(m|qr)$, while $m \neq q$, and $m \neq r$. Finally, if $F(b|qn)$ were true, then, since $F(m|qb)$, it follows that $F(m|qn)$, and hence, since $F(q|mn)$, it would follow that $q = m$, which is again false by construction.

Therefore, no one of the elements of the triad (q, b, n) is a mediator of the other pair. The conjugate resultant, *in this triad*, of the element n, is an element which is a mediator of the pair (q, b) (by 98); and hence, since, m, a mediator of (q, r), is the conjugate resultant of n in ρ, it is impossible that the resultants of the triad (q, b, n) exhaust the collection ρ. Meanwhile, the triad (q, b, n) possesses resultants which are not resultants of any one of the pairs (q, b), (b, n), (q, n). And so the triad (q, b, n) may be viewed as a two-dimensional complex.

It thus follows *both* that the resultants of ρ, taken as a whole, can be viewed as the resultants of a triad, if we choose, rather than as the resultants of a pair; and that triads such as (q, b, n) can be defined, in ρ, in such wise that a triad (q, b, n) possesses resultants which are not resultants of any of its single pairs, and which are still but a part of the resultants of the system ρ. Any such triad, however, may be viewed as a two-dimensional structure.

The viewing of ρ as a two-dimensional complex with reference to ρ_1 as a one-dimensional complex, is typical of a process which can be repeated any number of times. For, since ρ_1 is itself inclusive of ρ_2, etc., ρ_1 may be viewed, with reference to these included systems, as a complex possessing two, three, or n dimensions, where n is any whole number. According as this is done, ρ comes to be viewed, with reference to a particular series of included collections, as of three, four, or $n + 1$ dimensions.

The result of the foregoing considerations is that, within any portion of the system Σ which contains at least one pair of non-equivalent elements, we can define, pairs, triads, etc., in brief, collections of any finite number of mutually non-equivalent elements, such that, if such a collection, say α, possesses n elements, there exist resultants of the whole collection which are not resultants of any partial collection of the elements, containing only $n-1$, or $n-2$, or any less number of these elements themselves.

We may call the complexes of the resultants of such collections n-dimensional complexes. But it is observable that any such complex, once given, may also be treated, by the proper choice of conjugate resultants, as a complex of the resultants of a single pair, and so as a one dimensional complex. So that all such dimensionality is entirely relative to processes and structures of the type that we have just been defining.

124. Such structures become, however, of a more positive significance if we take account of the following application.

By a *line* shall be meant a structure of the general type of the chains of 83, only completed by the insertion of certain mediators. A line shall be a collection of elements such that in case of any triad of the elements of the collection, one member of this triad is the resultant of the pair composed of the other two.

And, in particular, the lines that we are here first and mainly to consider are to be subjected to the entirely arbitrary restriction (foreign to the first principles of our system Σ, but quite capable of being satisfied by a due selection of its elements as their existence has now been established), that if any two non-equivalent entities of a line are given, *no other line*, in the set of lines that we are to consider shall at once contain both of these elements.* In other expression, let the collections which are to be called lines be so selected that, if (a, b, c) is any triad of elements belonging to the same line, $F(abc)$ is true; while, if $F(apq)$ is true and $F(bpq)$ is true, and if at the same time $F(abp)$ is false, then we shall so select that $p = q$; so that if (a, p, q) is a triad of elements belonging to one of the lines now to be selected, while (b, p, q) is a triad belonging to another of these lines, and while (a, b, p) is no linear triad at all, then we shall be required so to select that $p = q$.

125. That selections of this sort are possible the theory of the chains, as developed in 83 sqq., has already shown. Such chains as were there defined might be constructed, as we now may observe, intersecting one another any number of times. For if (c, d) be any pair of elements belonging to a chain, the resultants of (c, d) form no single chain, but lie in sets subject to principle VI, which demands the existence of conjugate resultants, not only in the collection of the resultants of the pair (c, d) itself, but in every one of the countless collections of resultants of the pairs intermediate between c and d, as

* The development is here wholly due to KEMPE's initiative.

these pairs have been characterized, in their mutual relations, in 122, 123. It is possible, within the limits of any pair of non-equivalent elements (c, d), to define any number of segments, that is of intermediate pairs $(p_1, p_2), (p_2, p_3)$, (p_3, p_4), etc., each of which consists of mediators of (c, d), while all the elements concerned form triads such that $F(p_i p_j p_k)$ is true of any one of these triads. A chain, or rather a series of chains, can be run through such a series of intermediate pairs, according to any desired principle of selection from amongst the elements present in the various systems of resultants encountered. By virtue of the results stated in 117, a set of successive chains can be enlarged to a complete line, resembling perfectly, in its structure, a continuous geometrical line, by a mere insertion of intermediates and sets of intermediates. The special principle of selection assigned for the lines now to be considered will not only ensure that two lines have never more than one intersection, but in combination with the definition of a line will also exclude that degree of wealth of elements which forbids the arrangement of all the resultants of any pair of elements in Σ in a single linear serial order. For in the system Σ as a whole, if m and n are equivalent neither to a nor to b and are conjugate resultants of (a, b), $F(m|ab)$ and $F(n|ab)$ are both true; while $F(m|nb)$ and $F(n|mb)$ are both false. The principle laid down for the selection of line-collections will therefore forbid the inclusion in a given line of more than a single pair of conjugate resultants of any one pair. Thus, if c and d belong to the line, c and d themselves will be conjugate resultants of their own pair. And the intermediate elements of the line will be in Σ resultants of that pair. But no conjugate in (c, d) of any such resultant of (c, d) as belongs to the line, will lie in the line, except c and d themselves.

126. Since collections possessing the dimensional structure described in 123 exist in any region of Σ, it will always be possible to define systems of lines as follows: Let (d, e, f) be any triad such that it possesses resultants not contained amongst the resultants of $(d, e), (e, f), (d, f)$. If $(d, e), (e, f)$, (d, f), belong to lines that are amongst those lines which are here in question, and if x be any resultant of (d, e, f) which is not a resultant of (d, e), nor yet of (e, f), nor yet of (d, f), then it will be possible to regard (x, d) or (x, e), or x united with any element of the lines (d, e), etc., as constituting a new set of segments of lines. The result will be a two dimensional complex of elements. That method of construction of the n-dimensional collections or complexes of elements of Σ which has been indicated, enables us to regard these complexes, with all the lines, segments, etc., which are involved, as possessing an extent and variety of elements such as to permit us to define new sets of elements beyond any segments or bounded complexes once defined. Therefore, in selecting elements for our present purpose, we may regard these new elements as extensions of the lines and other complexes, while the dimensionality of the

complexes of lines which we may thus consider is subject altogether to our pleasure, under the conditions now in general laid down.

If, in consequence of the foregoing considerations, we compare the set of relations that we can thus define with the relations known to geometry, a natural method presents itself in the form of a juncture that may now readily be effected between our account and Dr. VEBLEN's *System of Axioms for Geometry*, (Transactions of the American Mathematical Society, July, 1904).

Dr. VEBLEN's expression "in the order ABC," becomes, in our terms, the assertion $F(b \mid ac)$. If we agree, in studying the constitution of our system of lines, to take explicit account only of non-equivalent elements, if we here call our elements *points*, and if we also adopt Dr. VEBLEN's definitions, his axioms appear in our statement as follows:

Dr. VEBLEN's first axiom covers our own principles III and IV, according to which our system contains a pair of elements. Axiom II of Dr. VEBLEN's set, interpreted in our terms, declares that if $F(b \mid ac)$, $F(b \mid ca)$. This needs for us, no comment. Axiom III asserts that, if $F(b \mid ac)$, then $F(c \mid ba)$ is false. Our own principles require that, in this case, $b = c$; and the axiom may therefore be regarded simply as excluding us from treating certain pairs of equivalent elements as distinct elements. This is merely a principle of selection. Axiom IV asserts that if $F(b \mid ac)$, then $a \neq c$. Our principles like KEMPE's, require that if $a = c$, $a = b = c$. Axiom IV, therefore, again excludes the regarding of certain equivalent elements as, for the present purpose, distinct. Axiom V of Dr. VEBLEN's set requires that if $a \neq b$, c exists such that $F(b \mid ac)$. This principle is provided for by our principles, which show that every pair defined in any of our sets of lines may be viewed as included in larger systems possessing linear F-relations. Axiom VI defines the important, but for us, quite arbitrary principle that governs the selection of the line-elements: "If points c, d $(c \neq d)$ lie on the line (a, b), then a lies on the line (c, d)." This agrees with our foregoing statement in 124. Only, with us, this is *merely* a principle of selection. Axioms VII, IX. relate to dimensionality, and demand points existent in triads and tetrads such as we have provided for in the foregoing. For us, such requirements are permitted by the system Σ in an infinity of ways.

Axiom VIII, the "triangle transversal" axiom runs, in our terms, thus: If the triad (a, b, c) is of the two-dimensional character described in 123, and if d and e exist such that $F(e \mid ac)$ and $F(c \mid bd)$, then f exists such that $F(f \mid ab)$, and $F(fed)$.

This axiom for us is, if we grant a certain mode of selection of elements, a theorem, resulting from the theorem of 99 — an incidental result, as it may be called, of the theory of conjugate resultants. That is, theorem 99 secures the existence of elements which may be selected so as to verify axiom VIII.

The theorem of 99 runs that if $F(\beta | \alpha)$, β and α have at least one resultant in common.

By the hypothesis of axiom VIII, $F(c | e\bar{a})$ and $F(c | bd)$. Hence $F(e\bar{a} | bd)$. Hence follows $F(e\bar{d} | ba)$. By 99, $(e, \bar{d})$ and (b, a) have in the system Σ at least one resultant in common (they have in fact, in Σ an infinite number in common). Call this resultant f. Then f exists such that $F(f | ab)$, and $F(f | e\bar{d})$; i. e., $F(e | fd)$ so that (at least) $F(fed)$.* That the common resultants here in question should belong as points to the system of lines that we have selected from the system Σ, is itself a matter of the mode of selection used. The properties of the system Σ simply insure the possibility of such a selection.

Axiom XI, Dr. VEBLEN's form of the postulate of continuity, is provided for by our own result, holding for the system Σ in general, stated in 117. This result ensures the possibility of the continuity of the line-collections, in case we choose to select suitable sets, precisely as the same result ensures in the system Σ as a whole, the existence of "products" and "sums." Axiom X, which limits Dr. VEBLEN's system to three dimensions, is for us a perfectly possible, but again quite arbitrary limitation; and the same can be said of the parallel line axiom XII, which concerns wholly the limitation of the selection of the lines admitted into a given system.

Our own "transversal" theorem, in 99, justifies, in terms of our principles, the remark made by Mr. KEMPE, upon the basis of his postulates, to the effect that any F-collection which contains a finite number n of elements that belong to the sets selected as the lines of the foregoing discussion, represents a definite configuration of points in a space of $n - 2$ dimensions.

Thus $F(abc)$ implies that (a, b, c) is a triad of points on one line. $F(ab | cd)$ is to be interpreted as follows: The pairs (a, b) and (c, d), lie by hypothesis, upon some selected pair of lines of our geometrical set. The problem is, how are these two lines to be related? The assertion $F(ab | cd)$ requires, by 99, that, in Σ, x should exist such that $F(ab | x)$ and $F(cd | x)$. If then x be viewed as one of the selected elements of the geometrical set in question, the assertion $F(ab | cd)$ may be viewed as the assertion that the lines through the segments (a, b) and (c, d) have in common a point of intersection which belongs to the mediators of (a, b) i. e., to the points of the segment (a, b), and also to the segment (c, d). On the other hand, if $F(ab | x)$ and $F(cd | x)$ are given, our principles require that $F(ab | cd)$. So that this form of assertion defines a pair of intersecting lines. The assertion $F(x | abc)$ defines a resultant of the triad (a, b, c). If this triad is to be viewed, in the way heretofore defined, as determining a complex of two dimensions, then x is a point

* KEMPE's theory is explicitly based upon two forms of the transversal theorem, assumed at the outset. For this our statement of the theory substitutes the postulated existence of conjugate pairs. What KEMPE sets at the beginning we thus reach at the end.

lying within the triangle (a, b, c). If $F(abc \mid def)$, wherein all the elements are mutually non-equivalent, and wherein each triad is to be viewed as a two-dimensional complex, then x exists common to these two areas, or two-dimensional complexes here in question. Thus all the intersection theorems of geometry may be stated in the form of the assertion of F-relations, with a due regard to the limitations of the classes of selected elements.

The principle of continuity is, for such a geometry, merely a principle of the selection of the elements, a principle which the system Σ permits, but does not require to be carried out.

Instead of such systems of lines as have here been selected, systems of lines whereof any two have two, three or n intersections, are perfectly permissible, so far as the system Σ is concerned. The possibility of a free, but definite variation of space-forms in a infinite number of ways, is thus provided for by the system Σ; and the outlook for a basis for generalized space-conceptions is all the more attractive, since the structure of the system Σ, based as it is upon fundamental logical principles, makes a test of the logical possibility of any proposed geometry a perfectly definite task — namely the task of seeing whether Σ actually contains complexes which are suitable to embody the desired space-form.

Since O-collections at once possess, as their resultants, all of the elements of Σ at once, no definite view of their dimensional structure is any longer possible. Hence selections suitable for space-forms must exclude O-collections; and so, as Kempe again points out, no geometrical set contains the obverse of any of the elements of the set. It follows that "spaces," defined in the foregoing way, always occur in Σ in pairs, such that to any one space-form σ there always corresponds a space-form, or collection $\bar{\sigma}$, constituted of the obverses of the elements of σ. These two space-forms are related, in Kempe's view, somewhat as two hemispheres.

Finally, since metrical relations can be reduced, in the known way, to ordinal relations, Kempe has briefly pointed out (as mentioned in the introduction to this paper), that sets of the elements of Σ can be so selected that operations corresponding to the addition and multiplication of the ordinary algebra of quantity, will enable us to select elements that may be viewed (with reference to certain arbitrarily assumed constant triads of reference-elements, i. e., bases), as the sums or as the products of given pairs of elements. Hence, *without introducing new elements*, the elements of Σ, if viewed in certain ways, enable us to define, not only the algebra of logic, but the algebra of quantity.

Note on the independence of the six principles.

That principle V is independent of the other principles is proved, in 84, by the assumption of a system Σ' consisting of a single pair of obverses.

That principle IV is independent is proved by the reasoning used in 35. For if we assume a system Σ'', *all* of whose monads and possible collections are to be defined as O-collections, while the system itself comprises any arbitrarily chosen number of elements, all the principles except IV and V are satisfied by the possibly existent collections of Σ'', while principle V is satisfied "vacuously," since no pair of non-equivalent elements exist. But in such a system principle IV is false; since all the elements are, by the definition of equivalence. mutually equivalent.

Principle III, and that principle alone, would be violated by an empty system; and that principle is therefore independent.

If, instead of the O-collections, we had used, as the basis of our account of the system Σ, the indeterminate F-collections of Σ, all the principles I, III, IV, V, VI would remain true if we viewed them as statements regarding indeterminate F-collections, and could therefore have been used as principles for the system of F-collections. But principle II is false if interpreted as applying to F-collections. For $F(xx)$ is always true, since $O(x\bar{x})$. But from $F(\eta x)$ does not follow $F(\eta)$. Hence principle II is independent of the other principles.

If we consider the class of those O-collections of the system Σ which are either pairs or triads, but which contain no greater number of elements than three, we may call this class the class of the O_1-collections. For this class of collections, principle I is false, since the O_1-triads cannot be enlarged, by the adjunction of any new members. In order to apply principle II, the hypothesis of that principle must be read as applying to a collection β which, in order that it should be an O_1-collection at all, must not exceed a triad. If $O_1(\beta)$, where β is a pair or a triad (so that $O(\beta)$ is also true), and if $O_1(\delta b_n)$ is true, where b_n is any one of the two (or three) elements of β, then δ itself (by the definition of the O_1-collections), cannot exceed a pair; otherwise (δ, b_n) would be a collection of more than three elements. Principle II then becomes equivalent to the assertion that, if β is a pair or a triad, and if δ is a monad or a pair, if $O(\beta)$ is true, and if $O(\delta b_n)$ is true, then $O(\delta)$ is true. Hence principle II is true of the O_1-collections. Principles III and IV are obviously true of the elements of Σ, considered with reference to the O_1-collections. The equivalences and non-equivalences are, in fact (because of what is proved in 27–30), unchanged by the limitation of our view to the set of O_1-collections, since all equivalences and non-equivalences are already concerned in determining the relations of obverses. And principles V and VI, which require the existence of certain O-triads (such as are also O_1-collections), remain true, although the hypothesis of principle VI becomes limited, in its application, to the mention of the complements of pairs and of monads. Thus principle I is independent of the other principles.

To prove the independence of principle VI:

Consider two pairs of mutually obverse collections of elements of Σ, viz., α and $\bar{\alpha}$, such that each is a line of elements, defined as follows:

(1) Every element a_n of α is a mediator of a pair (x, y) of elements of Σ, such that neither x nor y belongs to α, while $E(xy)$ is true. Thus $F(a_n|xy)$ is true of every element a_n of α.

(2) Whatever triad of elements of α be chosen, say the triad (a_m, a_n, a_r), $F(a_m a_n a_r)$ is true of this triad.

(3) Whatever element a_n of α is chosen, a_q and a_r exist, belonging to the collection α, and such that $F(a_q|xa_n)$ and $F(a_r|ya_n)$, while $a_q \neq a_n$, and $a_r \neq a_n$.

(4) No two distinct elements of α are mutually equivalent elements of Σ.

(5) Whatever pair (a_q, a_r) be chosen from amongst the elements of α, a_m exists such that $F(a_m|a_q a_r)$ is true.

From this definition of the line α, the properties of the obverse collection $\bar{\alpha}$ at once follow. The elements of $\bar{\alpha}$ are mediators of the pair $(\bar{x}, \bar{y})$. No element of $\bar{\alpha}$ can be equivalent to any element of α; for if any mediator of $(\bar{x}, y)$ is equivalent to a mediator of (x, y), then $O(xy)$ is true, which opposes condition (1). If then c and d be distinct elements, chosen in any way from the total collection $(\alpha, \bar{\alpha})$, $c \neq d$ by construction.

Whatever pair (c, d) of elements of $(\alpha, \bar{\alpha})$ be chosen, it follows that there exists g, also belonging to $(\alpha, \bar{\alpha})$, and such that $g \neq c$, $g \neq d$, and $F(g|cd)$. For if c and d both belong to α, the existence of g, as an element of α, follows directly from condition (5). If c and d be both chosen from $\bar{\alpha}$, a precisely analogous result holds true. But if c be chosen at random from α, and d from $\bar{\alpha}$, then let $c = a_n$, and let $d = \bar{a}_q$, where $\bar{a}_q$ is such that its obverse a_q is such that $F(a_q|xa_n)$. Hereupon, choose in α an element a_t such that $F(a_t|a_q x)$. This, by condition (3), is always possible. Since $F(a_q|xa_n)$ and $F(a_t|xa_q)$, it follows that $F(a_t|a_q \bar{a}_n)$ is true; hence $F(\bar{a}_t|\bar{a}_q a_n)$ is true; and thus, if $g = \bar{a}_t$, $F(g|cd)$ is true of an element g which belongs to $(\alpha, \bar{\alpha})$. Similarly, were it true that $c = a_n$, and $d = \bar{a}_r$, where $\bar{a}_r$ is an element such that $F(a_r|ya_n)$, then we might choose a_w such that $F(a_w|ya_r)$. This, by condition (3), is also possible. Eliminating y we have $F(a_w|a_r \bar{a}_n)$; i. e., $F(\bar{a}_w|\bar{a}_r a_n)$; so that, if $g = \bar{a}_w$, we again have g belonging to the total collection $(\alpha, \bar{\alpha})$, and such that $F(g|cd)$ is true. Since all pairs of elements (c, d) thus chosen from $(\alpha, \bar{\alpha})$ are, by construction pairs of mutually non-equivalent elements; since g always exists, belonging to $(\alpha, \bar{\alpha})$ and such that $F(g|cd)$ is true; and since $\bar{g}$ also belongs to $(\alpha, \bar{\alpha})$; it follows that the collection $(\alpha, \bar{\alpha})$ contains sufficient elements to satisfy, with respect to any pair of elements of $(\alpha, \bar{\alpha})$, the demands of principle V, without going beyond the elements of this collection $(\alpha, \bar{\alpha})$ itself. That is, whatever pair of elements of $(\alpha, \bar{\alpha})$ be chosen, an element of $(\bar{\alpha}, \alpha)$ exists which is no obverse of either of the elements of the pair, and which forms an O-triad when adjoined to the pair.

Hereupon let us consider a system Σ_a, which shall consist solely of the elements of the collection $(\alpha, \bar{\alpha})$. Let there be formed, of the elements of this system Σ_a, collections which we shall call the O_a-collections. These O_a-collections shall be identical with those collections of the elements of $(\alpha, \bar{\alpha})$ which are O-collections in Σ. The O_a-collections of the system Σ_a will now conform, by construction, to principles I–V of the system Σ.

But, in case of the O_a-collections, principle VI will be violated. For (returning to the system Σ) consider any element a_n of α; and consider the totality of those elements a_q, belonging to α, and such that $F(a_q|xa_n)$ is true in respect of the system Σ. Call this totality the collection α'; so that α' is the collection of those elements of α which, in Σ, are mediators between a_n and x. By condition (1), x itself does not belong to α, and so does not belong to α'. Now, in the system Σ, the element a_n, since it belongs, by construction, to the collection α', possesses a conjugate resultant with respect to α'. Let r be this resultant, so that, in Σ, $\jmath_r(a_n r;\ \alpha')$ is true. It is manifest that r is an element such that $F(r|a_n x)$. Yet r is no element of the collection α' or of the total collection $(\alpha, \bar{\alpha})$. For if r were an element of α, then, by conditions (3) and (4), a_k would exist, belonging to α, and such that $F(a_k|rx)$, while $a_k \neq r$, and $a_k \neq x$. But a_k would also belong to α', and by the definition of a conjugate resultant $F(a_k|ra_n)$ would be true. Now $F(r|a_n x)$ is true; and from $F(a_k|rx)$ and $F(r|a_n x)$ follows $F(r|a_k a_n)$. If, at the same time, $F(a_k|ra_n)$, $a_k = r$; which is contrary to the hypothesis. Thus r is no element of α, and also cannot be any element of $\bar{\alpha}$. For since $F(r|a_n x)$, if there existed $\bar{a}_q$ such that $r = \bar{a}_q$, $F(\bar{a}_q|a_n x)$, and hence $F(\bar{a}_q|xy)$, would be true of some element a_q of which $F(a_q|xy)$ would also be true. In that case $O(xy)$, which is contrary to condition (1).

Since r does not belong to α nor yet to $\bar{\alpha}$, r does not exist in the system Σ_a. There is then, in Σ_a, no element capable of meeting the requirements of principle VI as applied, in this system Σ_a, to the partial collection α'. So principle VI fails in Σ_a; and is therefore independent of the other principles.

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Editor's Note: This essay was reviewed in the *Journal of Philosophy*, Vol. III, pp. 357-361, by Theodore de Laguna. Professor C. I. Lewis explains the significance of the theories of Kempe and Royce in *A Survey of Symbolic Logic*, pp. 362-370, and in the *Philosophical Review*, Vol. XXV, pp. 407-419.

See also the interesting controversy on the theory of mathematical form between Charles S. Peirce and A. B. Kempe in the *Monist*, Vol. VII, pp. 178ff., and 453ff. Peirce criticized Kempe for defining a relationship as "nothing but a complex of bare connexions of pairs of objects." Kempe replied: "I have never held or expressed, either directly or by implication, any such opinion as he attributes to me." (P. 452.) Peirce refers to Kempe's "Memoir on the Theory of Mathematical Form" as a "profound and masterly" treatise.

INDEX

U

V

W

Y

Z