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Lecture IV

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Lecture 4.

[43] The summary of the various types of scientific concepts which we have given in the former lectures now needs to be supplemented by some further applications to conception that are well known to be of general use in science. Our assertion has been that in terms of classes, relations, ordered series, levels and transformations, we are able to define not only the various conceptual forms which appear in science but the properties and combinations of those conceptual forms which are of the utmost service in the work of science. Anyone who has followed our discussion will also be no doubt in the mind of anyone who has considered the illustrations used in our foregoing discussion that ordered series, that what we have called “levels” and what we have defined as “transformations”, can be, and very often are, part not only of the work of science but of the applications of human thought. But someone who has listened to the foregoing discussion may have missed the express mention of certain concepts or conceptual forms that are universally recognized to be of great scientific importance. “What,” one may ask, “has been said with regard to the significance of quantitative conception for exact science?” When in some of the natural discussions of the ideals and methods of science it was asserted that every science is a science in so far as it reaches the stage where the application of mathematical formulae is permissible and fruitful, those who made use of such expressions generally added the definition that mathematics is the science which deals with quantities and their relations. From [44] this point of view the concept of quantity especially marks the line where mathematical and exact procedure begin to be possible, and consequently the concept of quantity would occupy a place in the rest of human concepts, that ought not to be ignored in any statement of the characteristic concepts of science.

As a fact, however, the foregoing definition of the scope of mathematical science is no longer adopted by those who are acquainted with the modern advances of mathematics. Mathematics is certainly not exclusively the science of quantity. It would be nearer the truth to say that mathematics is the science of exactly definable order. And as we shall see in a moment, this definition would include within its scope the whole field covered by the older definition of mathematics. It is no part of my purpose, however, to defend even that definition of mathematics as quite sufficient to cover the whole scope of that science. It does concern us, however, to note that there are unquestionably mathematical sciences which have no occasion to make use of quantitative concepts. The best known and the most highly developed of these branches of mathematics is modern "projective geometry", which deals with systems of space relations in a precise way, in fact a way that can be stated almost wholly in symbolical language, while on the other hand the projective relations are very sharply distinguishable from what are called metrical relations of space, and can be treated without using a single concept of a quantitative nature. The modern science of what has been called the "algebra of logic", the discussion namely in symbolical language of the fundamental relations of logical classes, and of the properties of systems of relations, is a mathematical science but in no sense [45] a science that directly depends upon dealing with quantities. If one is therefore to maintain from the modern point of view that a science becomes exact in proportion as it becomes mathematical, one must modify what used to be regarded as the application of such of definition. A science becomes mathematical when you give an exact account of the fundamental principles which determine very vast ranges of fact in that science, and when you can deduce from your statements of the fundamental principles the precise results by methods such as those of algebra, or of any similarly precise construction of symbolic systems. The prominence of the idea of quantity in the older mathematical science, was simply due to the fact that quantities are objects whose system of relations is peculiarly fit to be treated in mathematical language.

It is my present purpose to point out in the next section of this discussion, the reason why quantity has proved to be so an important an object for exact research, and why the quantitative conceptions are indeed of such significance in many exact sciences.

The study of our ideas of quantity has been very profoundly revolutionized so far as the fundamental logic of the topic is concerned, by the mathematical tendencies of the last fifty years. Whatever may be one's system of philosophy, it will henceforth be impossible for those who are acquainted with modern results to regard what has been called the "category of quantity" as a fundamental and simple, or as an indefinable category of science. When we call a set of objects quantities, we do so for reasons, some of which have possibly not yet received their full and final [46] definition, while some of these reasons can be very precisely stated. Unfortunately in dealing with the subject, often decidedly modern writers who are aware of the recent results vary a great deal in the vocabulary they employ, so that an effort to follow the literature of the subject is decidedly confusing to the elementary student. It may be hoped that the new encyclopedia of mathematics now in process of publication in Germany will do much to help towards a clarification not only of ideas but of terminology regarding that subject. Russell in his remarkable, but not altogether perspicuous treatment of the topic in his recent work on the principles of mathematics has suggested some important, but I think, not altogether defensible changes of terminology. Still, on the whole, in the present discussion I shall try in a measure to follow a terminology somewhat like that of Russell.

It may first be well to distinguish between what we may call quantities in a general sense, and what we may call magnitudes, using the latter word in a somewhat restricted meaning. I may take a quantity of water, a quantity of iron, and a quantity of any other substance you please. I may compare these various quantities as to their weight. The mass, namely, of matter can be studied, without regard to the physical or to the chemical constitution of the matter that is in question. Suppose that the quantities of iron, etc., that are in question, prove to be, when measured with respect to mass, equal. In such a case as that we should frequently say that these masses of matter were equal in weight or in mass, according as we laid stress upon the more fundamental notion of mass, or upon the concept practically sufficient at any one place [47] and time, the concept of weight. If we shoes to define our terms a little more precisely we might say that the mass or the weight of these various quantities of matter is of the same magnitude. In this case we should use "magnitude" as the name for the character in respect to which all the various quantities were said to be equal. The quantities themselves would be placed on a level. What level they were placed upon would be determined by their common magnitude. But we may agree that any magnitude of a given kind shall be regarded as something unique, so that it could differ from another magnitude of the same kind, only by being greater or less than that magnitude. From this point of view we should say that the quantities have the same magnitude. But the quantities themselves might be various quantities that were equivalent to one another. This, as I understand him, is substantially the usage that Russell proposes. Any particular kind of magnitudes, as for instance masses or weights, would form a single ordinal series. Each one of these magnitudes, for instance the magnitude that stands for one hundred unites [sic] of mass, would define a level. All the quantities whose magnitude was one hundred units, would now lie upon this same level and would be equal to one another. Thus all the quantities having the same magnitude would form a class of objects upon a certain level. But we should not say that two magnitudes are equal to each other; but only that two quantities have the same magnitude.

If for the time we agree upon this usage, which I need not tell you is not universal in the literature on the subject, we have before us two questions relating to the nature of quantity and magnitude. Let us suppose that we have a great number of lines [48] which agree in length. The length of all these lines maybe regarded as a magnitude, in so far as that length is supposed to be a character that all these lines have in common. But when we compare the lengths of two lines together and consider these lengths as belongs each to its own line but being equivalent each to the other, then we treat the lengths as quantities. A hundred feet or meters regarded as a magnitude would be a determinate and unique length, such that all equal lines, each one of which was one hundred meters long or one hundred feet long would agree in having that same magnitude. For each different kind of quantities, for instance for masses of material substances, for lengths, for temporal durations or for any other kind of quantitative objects, there would be a separate series of magnitudes. But no set or kind of quantities would permit various series of magnitudes. All quantities of the same kind could be according to this usage compared together. They would either then agree in magnitude, when we should say they were of the same magnitude; or they would disagree, and then we could say the magnitude of one of them was greater or less than the magnitude of the other. The quantities would be definable as objects that can possess magnitude. The magnitudes themselves would be defined as objects that could be arranged in series, which for the first would be defined in terms of the well-known relation of greater and less. Considering length as a magnitude which various lines have in could have in common, on length would differ from another length as a magnitude being greater or less than that length.

If with this distinction between quantity and magnitude in mind we proceed to study various magnitudes, we soon find forced [49] upon our attention the two types of magnitude which have made so much trouble for the students of the logic of science. Some magnitudes are of the character of intensive magnitudes, or as we often say, intensities. Some magnitudes are of the character of extensive magnitudes. A typical intensive magnitude is temperature. A typical extensive magnitude is length. The magnitudes with which the psychologist has to deal appear to be intensive magnitudes. For instance, the magnitude of a pleasure or of a pain, the characteristic which leads one to say that one sound is louder than another, one pressure is more intense than another, these are intensive magnitudes. Mass, on the other hand, is another classic instance of the extensive magnitude. A very natural question arises as to what is the real difference between intensive and extensive magnitudes. I will not attempt in this brief summary to indicate to you what various opinions have been held upon this subject. Perhaps the most customary account of the difference, which you find in philosophical discussions, is the statement that "extensive magnitudes consist of parts, while intensive magnitudes do no consist of parts." Some writers add that "an intensive magnitude does consist of parts, but that these cannot be separated by us, while the parts of an extensive magnitude can be separated by us." I can divide a mass of matter, so as the show that the greater mass consists of lesser masses. But I cannot divide a great pleasure into parts, or a great pain into constituents so as to show that the whole is made up of these parts. "Yet," say the writers in question, "even in the intensive magnitude the parts must be there." Sometimes the nature of an intensive magnitude is defined in these terms, "that in order to produce a given intensive [50] magnitude, you must pass through various stages, leading as it were from zero to this intensity of degree. On the other hand, when you have to obtain a given extensive magnitude, its is said that you must put parts together in order to compose the greater magnitude out of these lesser parts.

Now that to which I wish to call your attention in this discussion is the fact that the concept of magnitude, both in its intensive and in its extensive form, is a particular case of the concept of a collection of objects that can be arranged in an orderly series. The most important aspect of the magnitude is after all their serial character. For the whole study of the logic of modern mathematics seems to indicate that what makes magnitudes so important for the purposes of exact sciences is the fact that its properties admit of a strictly orderly distribution. A series of magnitudes is an ordered series. And that is why we can correlate a series of magnitudes to our series of numbers, whole and fractional, or on occasion rational and irrational. This correlation of the two series is distinctly an ordinal correlation. The numbers are not, as some have supposed, logically mere names, which have been devised by the human mind for characterizing quantities or even magnitudes. On the contrary, numbers have a character, which makes their most important properties logically independent of their relation either to magnitudes or to quantities. All the fundamental laws of arithmetic and of algebra can be developed without making any reference whatever to the extent of objects that resemble either the physical quantities that we equal or unequal, or the magnitudes in respect to which we define the quantities as equal or as unequal. The logical possibility of a reduction of the entire science of arithmetic to the [51] theory of certain ordered series, is one of the most remarkable results of recent logical investigation. I can here report only the general nature of this result, but as some faint notion of it is necessary to a proper comprehension of the concept of magnitude of [sic] the distinction between intensive and extensive magnitudes, I must turn aside for the moment from speaking of the magnitudes and mention the still more fundamental conception of the numbers.

Nothing is more familiar to us than the concept of number. There can be no doubt that the concept of number, as we actually find it in use, has sprung through successive elaborations from the concept of the whole numbers. To the whole numbers advancing arithmetical science early added fraction. The discovery of the existence of irrational relations amongst magnitudes early led to the general conception of what the Greeks called the “irrational” without yet defining it as an irrational number. In modern arithmetic the irrational numbers were early introduced, although their logical nature received no definition until about thirty years ago, when three different mathematicians within a short time were led independently to an exact definition of what was meant by an irrational. Now there is no doubt that this development of the concept of number from the concept of a whole number through the concept of the rational fraction, to the concept of the irrational number, was determined by the use that early made numbers the expression, first of certain characters belonging to collections of objects, and then of certain characters to which attention was attracted by processes of measurement. In other words, if the human mind had not had to deal with magnitudes, the number system would [52] never have received the development that it did receive. On the other hand, at every stage of this development the logical characters which numbers, whether whole, fractional, or irrational, obtained, were throughout essentially ordinal characters. It is perfectly true that I can define the so-called “cardinal” numbers without reference to the concept of ordinal numbers. On the other hand, it seems to be perfectly true that I cannot define the use of number for the purposes of aiding in the measurement of magnitudes without laying an almost exclusive stress upon the ordinal properties of our various number series.

Let me speak briefly of what the just mentioned ordinal characters and numbers are. All the whole numbers form an ordered series. It is in some respects the simplest of all the ordered series. How we came to conceive of such a series need not at this moment concern us. What is certain is that the concept is natural and was extremely familiar to the human mind from a very early stage. What is also certain is that the concept is natural and was extremely familiar to the human mind from a very early stage. What is also certain is that you can deduce all the properties of an ordinal number series without even mentioning the existence of a magnitude, you can deduce, I mean, all these properties from certain fundamental propositions which require for their understanding only the first principles of the logic of classes and relations.

At all events, now that we have the concept of the whole number series, the constitution of this concept is simply what follows: The whole number series is a collection of objects in which there is a first term, and a certain intransitive non-symmetrical relation, which we may call the relation <u>next successor of</u>. Now the first number has a successor. Every number has [53] No [sic] two numbers have the same <u>next successor</u>. Every whole number except the first is the <u>next successor</u> of some whole number. There is no last number, just because every whole number has a "next successor." Now these are the fundamental properties of a whole number series, and they are purely ordinal properties. The ordinal series that thus results conforms to the general law of all ordered series which I mentioned at the last time. Although the relation <u>next successor of</u> is an intransitive relation, it leads us to be able to define the general relation "successor of", the relation in which any later number stands to any earlier number. This relation is transitive. Of any two whole numbers one is "successor of" the other. There is a whole number namely the first, Number one, which is the "successor" of no number. All the other numbers are in a transitive sense "successors" of one. And so much for the moment for whole numbers. In the course of the development of mathematics this whole number series has been supplemented as I have said, by several additions to the number concept which is produced by joining to the whole numbers the negative numbers. The negative numbers form an inverted series of the type of the whole numbers. They may be developed out of a concept of the whole number by assigning to the first number, unity, a predecessor zero, and then by permitting zero to have a predecessor – minus one, and by thus developing a series of predecessors of the natural whole numbers. But more directly connected with the concept of magnitude, in the ordinary usage of that term, in which only positive magnitudes are in question, is the conception of the rational fractions. The rational fractions have properties which people are accustomed to deduce from a previous knowledge of certain [54] relations of magnitudes. But such a deduction is quite unnecessary, for the logical development of all properties of the rational fractions, considered as numbers. The rational fractions form an ordinal series. If you consider the rational fractions which are greater than zero, and if you suppose all these rational fractions to be reduced to their lowest terms, and if you suppose a rational fraction that is equivalent to a whole number to take the place of that number, you then get a series of definable objects, such as ½, ¾, 2/2, etc. This series of ideal objects has no first term, and no last term. It has the notable law that between any two terms of the series there is always another, so that in consequence between any two terms of the series there is always an infinite number of others. The ordinal relations of the rational fractions are determined by a very simple rule. If you wish to find which one of two rational fractions comes earlier in the ordinal series, or is in ordinary parlance less than that ordinary rational fraction, you perform the operation of reducing the two to a common denominator and of then considering which of the two we reduced has the less whole number for its numerator. By means of this simple transformation the ordinal relations of any two rational fractions, and consequently of each rational fraction, whole series of rational fractions can be precisely determined.

There is no time on the present occasion to define or to describe in what way the concept of the rational numbers can be joined to the concept of the so-called real numbers, a concept which includes such objects as the “square root of two” and the “logarithm of fifty”. It is enough that when the concept of number is joined in such wise that we are able to define the series of all the [55] rational numbers between zero and infinity, the joined number series still loses none of the characters of the ordered series. Upon the conception of the ordered series of objects thus formed, the whole of arithmetic and algebra can be founded. Consequently, although it is true that the conceptions of these ordered series of numbers first come to mind in our efforts to deal with the measurement of magnitudes, the numbers themselves have logical properties that we can deduce without the lest reference to the system of magnitudes. In the ordered system of numbers we define as the characteristic transitive and non-symmetrical relation according to which the series is ordered, the relation of greater and less, a relation for which later and earlier might be substituted without any alteration of the logical character of the series that we are considering. In connection with the definition of numbers, an exact definition of the operations of addition, multiplication, subtraction, division, is perfectly possible. These various operations are transformations whereby from two numbers you can according to a fixed rule deduce another number which is their sum, or their product, as the case may be; or in the case of the converse operations, their difference or their quotient. And these operations can be defined still without any reference to the conception of magnitude. So much for the numbers. Now let us turn back to those realms of our experience in which we meet with the conception of magnitude.

There are regions of our experience, regions of the most various character, in which we meet a relation that we learn to define as the relation of greater and less. The relation between two noises or two pains, the relation between two temperatures, [56] the relations between two lengths, the relations between two masses, such relations are matters of experience. That such relations are possible we cannot attempt to deduce logically. But we can observe that the greater and less relation, wherever it is found, has the character of being a non-symmetrical and transitive relation. Now it happens that in a great many regions of our experience we meet with differences of greater and less, in cases where between any two objects, one of which is greater and the other less, we can either define, or for some reason become disposed to define, a postulate, a fact, that is related to both these facts in such wise that it is less than the greater of the two and greater than the less of the two. That is, if we have distinguished the length of two lines we can define a length greater than the less and less than the greater. In a great number of instances reasons upon which one can lay no stress lead us to postulate the existence of such a length. It is further true that certain properties of certain objects which possess magnitude, namely such properties as come to our attention in the course of ordinary geometry when we compare the diagonal of a square and the side of a square, lead us to the definition of magnitudes whose relation to other magnitudes is still more complex than the relation which has just been suggested. The world of magnitudes, then, gives us the relation of greater and less in extremely various ways. But when we are dealing with quantities of the same kind, with lines that have length, with quantities of matter that have mass, or volume, or any other such magnitude, we are led to arrange all the magnitudes of a given kind with which we are concerned in a single ordered series, and we now discover that we can by means of certain [57] conventions, which actually vary with the sorts of magnitude that we are dealing with, we can, I say, bring the order series into correlation with this series of magnitudes. That is, we can attach a number to a magnitude. And we can do so in ways that enable us to define very exactly what number ought to be attached to a given magnitude of a given series. Since the ordered series of the magnitudes has in so far the properties of the ordinal number series, since we can substitute the number series for magnitude and read back a result which we have obtained from considering the numbers into series of magnitudes, so as to use our arithmetic for purposes of defining or predicting results which held true for the various series of magnitudes, we accustomed to treat magnitudes on the one hand as if they were numbers, or on the other hand to regard the numbers as if they were merely names for magnitudes. But as a fact, magnitudes such as lengths, volumes, intensities, and so on, are given in experience and are not constructed out of our own logical conception. In other words, numbers, as mathematical science defines them, can be subjected to a purely conceptual process which needs no external experience to give it its precision of definition. A magnitude, then, is a series of ideal objects whose characters are suggested to us when we compare together such quantities as lines having length, pleasures having intensity, matters having mass, and so on. These ideal objects are primarily characters that we find to be the same in various quantities, as when two lines are the same in length. When we once obtain an experience of these characters we can abstract them from the quantities with which they were in the first place associated, and then arrange all the magnitudes of a given kind in [58] a single ordered series. To this series we can attach our numbers, and consequently make use of arithmetical results for the purpose of defining and predicting the properties of magnitudes. The character of the magnitudes which permits us to do this, is in the first place precisely their character as forming an ordered series. The character of the quantities, that is, of the concrete objects which appear in our experience and which we compare together, the character of these quantities, I say, which permit us to declare then to be the same in magnitude, are the characters which we have to define in terms of a great number of different natural and artificial processes, processes of science of measurement. But when we once carry out these processes we become convinced that quantities of the same kind may agree in having the same magnitude, or again may differ from one another in magnitude; so that thus we are led by determinate processes from the concrete facts of experience to the conception of the ordered series of magnitudes of a given kind. The logical value, however, of studying quantities in this way, and of arranging magnitudes in this relation to the numbers, is the value derived from the inexhaustibly wealthy properties of an ordered series, properties whose wealth the whole of arithmetical and ordinary algebraic science is engaged in developing. So much for magnitudes in general, for their relation to experience on the one hand and to ordered series on the other.

But now for the difference between extensive magnitudes and intensive magnitudes. The difference is unquestionably a difficult one. No exhaustive list of the various types of intensive [59] magnitudes has yet been made. No completely logical survey of the field is therefore yet possible. But on the other hand we must not make unnecessary mysteries with regard to the extensive and intensive magnitudes. Let us begin with the extensive magnitudes, because they are the best known, and from the very nature of their characteristic properties the most manageable. Extensive magnitudes are simply magnitudes whose properties permit of description in terms that enable us to use such operations as addition and multiplication. Of these relations the operation of addition is most fundamental. It is exemplified in the world of magnitudes by the operation of adding lengths of lines. To add the lengths of two lines is to carry out a transformation, which can be exemplified by any line you please, and by the process of adding to a certain stretch that has once been measured on this line another stretch or segment.

That this operation produces important results is a matter of experience. The operation in question precisely corresponds in its empirical characters to the characters which are possessed by the mathematical or arithmetical transformation known as addition. Two of the most essential characters of addition are these: (1) If “a” is a magnitude which permits of addition, and “b” is another such magnitude, then the operation of addition is such that “a+b” is the same magnitude as “b+a”. (2) If “a”, “b” and “c” are three magnitudes of the same kind which admit of addition, then if you first add “b” and “c” and add the result to “a”, the result is the same magnitude that you would produce if you first added “a” and “b” and then added “c” to the result. In the process of the addition of numbers, a process that, as I said, is definable in wholly mathematical, in purely logical terms, these same properties hold for the operation [60] in question. That an operation having these properties holds good when you deal with lengths, masses, and such objects, is a fact of experience. Helmholtz in his introduction of theories of physics, has insisted on the importance of the operation of addition for a very large number of physical processes, and has also insisted that only experience can determine whether an operation of addition is possible. What you mean by saying that, for instance, the addition of weights is possible, is simply this:
(1) That if you are dealing with weights, by putting first one of them into a scale pan and then another and then perhaps still another, you discover that in all such operations the result is the same whether you first put “a” into the pan and then “b’, or whether you first put “b” into the pan and then “a”. You discover that if you first combine the weights “a” and “b” and then add the weight “c”, the result is the same as if you had first combined the weights “b” and “c” and then added the weight “a”. But now it is not at all self-evident that such should be the result of weighing. If you are dealing instead of weights, with chemical combinations, the reaction produced by your combination might decidedly depend upon the order in which the constituents

were united. If you are dealing with social processes, as for instance with the addition of new members to clubs, it may make a great difference whether you first elect “a” and afterwards “b” or whether you first elect “b” to membership and afterwards “a”, or whether “b” and “c” first converse together and then “a” joins the conversing group, or whether “a” and “b” first converse together and then “c” joins the conversing group. In other words, while the results of an operation of addition when you deal with the logical objects called “numbers,” are perfectly determinate and necessary, from the relation of the [61] ordered system of objects called “numbers”, on the other hand, whether what amounts to an addition operation in dealing with physical objects is possible or not is wholly a matter of experience. Now as a fact, we have in the world great numbers of quantities which so behave in their relations to one another that the magnitudes each time in question are subject to the laws and operations of addition. Quantities of this kind may be called extensive magnitudes. And that seems to be all that we mean by extensive magnitudes.

As for the conventional definition before mentioned, that extensive magnitudes are magnitudes which consist of parts, and so consist of them that the parts can be distinguished by our observation or attention, this seems to be but an awkward and inexact way of saying that extensive magnitudes permit of the operation of addition. Any group of objects consists of parts, namely the objects grouped. But as we have seen in dealing with relations in general, and as we have still more seen in mentioning the social and moral relations, the fact that we can distinguish the parts of a group does not at all assure us that the parts are related to the whole in the fashion which is so characteristic of extensive magnitude. In other words, extensive magnitudes are not such because they have parts but because they are subject to the operation of addition. We may, if we like, express this by saying that in an extensive magnitude the whole appears as the sum of its parts; whereas in a social group, for instance in a club, the whole is not the sum of its parts. But this is only another way of saying that an operation having the properties of addition does not apply to any groups but to such as those that appear in the world, of the numbers on [62] the one hand, and of the extensive magnitudes on the other.

But the intensive magnitudes, what of them? “Why,” I reply, “they can be arranged in ordered series, in fact they demand such arrangement. They are subject, they exemplify the relation of greater and less. This relation as it appears in the intensive magnitudes is an empirical relation. You have to find out in each case what it means by experience. That one temperature is greater than another, that one pain is greater than another, that one friend is dearer than another, that one society is better organized than another, that any other difference of so-called degree is found in experience, that you must test each time by examination. Nevertheless, all such cases where degrees are found, are cases where the relation of greater and less has definite meaning. But when you deal with magnitudes that form what I may call a degree series, all the characters of an ordinal series prevail, so far as that is merely an ordinal series. But to the operation of addition which we can define in the case of the ideal ordinal series of the numbers, and which we can there find to possess arithmetically interesting results, there corresponds nothing that we can any longer verify in our experience of the intensive magnitudes. For instance, temperatures, as Ostwald points out, do not admit of an addition operation, that has a definite physical meaning. When the temperature rises from freezing point to thirty degrees above, you may say if you please that thirty degrees of temperature have been added. But what this physically means with respect to the extensive quantity called heat energy present in the system whose temperature you are measuring, that cannot be detected merely by measuring this difference in temperature. If the temperature rises, heat is added. Now a quantity of heat in a given system has a measurable extensive magnitude. But on the other hand temperature [63] in the system is not an extensive magnitude, because to speak of a rising temperature as an addition of more temperature tells you nothing definable about the physical properties of the system. Precisely so if we speak of the intensive magnitude of a pain. It is indeed true that to say this pain is greater than another has a perfectly definite meaning, where we have an opportunity for direct comparison of the two. On the other hand, to say that one pain is twice as great as another, or that a pain measuring three units added to a pain measuring two units would inevitably constitute a pain measuring five units, to say all that is to say something that has no verifiable psychological equivalent in our experience. Pains are intensities merely because they do not furnish to you facts that correspond to a definite addition operation. This, as I take it, is the essential character which distinguishes intensive from extensive quantities.

In the case both of the intensive and the extensive magnitudes, the significance of the concept for exact science primarily depends upon the fact that magnitudes form an ordered series. Now all the reasons that give mathematical importance to the various sets of transformations called additions, multiplications, and so on, make it very significant that we should be able to find in our experience objects which can not only be correlated with the number series, but which possess in addition significant properties that correspond to the operations of addition, and to related operations. Hence, extensive quantities are much more important, and their extensive magnitudes are much more significant for exact science than are the intensive magnitudes. If we add that intensive magnitudes can in general only be very imperfectly [64] correlated with the number series, while extensive magnitudes, just because the operation of addition make so many tests of our theories possible, can have their correlation with the whole numbers very accurately examined, and very elaborately corrected, we see why the extensive magnitudes, and the quantities whose relations are defined in terms of extensive magnitudes, form systems of objects in dealing with which our thought can attain some of its most marvellous [sic] successes in its struggle with external experience and in its effort to control nature. A word is still needed as to the relation of the various quantities which possess given magnitudes, to one another. In the usage that we have in this discussion adopted, a quantity is a name for a character possessed by a certain concrete object, say a line having length. Two quantities can be compared when they are quantities of the same kind. If they precisely agree, we say that they have the same magnitude. We also say that in respect to one another they are equivalent or equal. A set of equal quantities, that is quantities having the same magnitude, will form what in our former discussion we called a "level". Our devices for discovering whether certain quantities are equal are therefore devices for taking a level. Now it is customary to suppose that these devices are determined by absolutely obvious and perfectly necessary considerations, so that while doubt can arise about the accuracy of a particular measurement there can be no doubt whatever as to what the principles of measurement in any science where there are quantities should be. For measurement is of course a process of comparing quantities to see whether or no they have the same magnitude, and it therefore consists of a set of processes which are founded upon such comparisons, [65] when taken in combination with the operation of addition and its connected operations. Hence, measurement is possible with definite results, only in the field of the extensive quantities. But as I said, it is popularly supposed that what happens when we compare two extensive quantities is determined entirely apart from any convention by the nature of these quantities themselves.

However, a glance at the varieties of principles that pertain in the exact sciences of measurement, shows us that considerable doubt may exist not only as to the accuracy of particular measurements, but as to the nature of the standard in terms of which we define an equality. Especially, for instance as the definition of equal intervals of time lend to a great difference of opinion. The modern study of the logic of metrical geometry has shown that the definition of one length and of equal lengths, is a definition depending upon postulates that cannot be called self-evident, although our experience of space relations warrants us in saying that these postulates correspond with this experience. As a fact, what the relation between two quantities of a given kind should be, depends on the relation of equivalence, in other words, the problem [of] what the test of equality should be is a problem of no one answer. Experience of the various sciences determines, here as in other cases of taking levels, what the relation of equality shall in any case be declared to be. What is logically necessary, for the definition of an ordinary equality amongst quantities, is this: We must choose some symmetrical and transitive relation, whose presence admits of being tested by some kind of comparison such as can be exactly made, and such that the result is not ambiguous. Beyond this, the nature of the equality [66] depends largely upon definition. Thus, two segments of straight lines, considered apart from direction, are declared equal by the well known tests of possible superposition or congruence, but if you are dealing with directed lines, as for instances, with the so-called “vectors,” equality is not tested merely by congruence but by sameness of direction. And tests differ still more widely in the various sciences.

This summary sketch of the logical theory of extensive and intensive quantities and magnitudes has been intended to illustrate the way in which the serial concepts and the concepts of transformations and of levels find their application to the quantitative sciences. The result of our investigation is that the ideal exact science is not in any way peculiarly dependent upon the concept of quantity, but on the other hand that the concept of quantity derives its peculiar importance for exact science from tis intimate relation to the series and levels, which are definable in terms of the concept of number. The concept of number in its turn is simply the most highly developed concept, which ordinary physical science has occasion to use, and which at the same time is a concept of an ordered series. We have devoted most of our attention to that aspect of scientific concepts which is definable in terms of a one-dimensional series such as the number series itself is. A glance at the range of the sciences such as geometry, shows that a great deal of our exact science depends upon defining ordered systems that have two, three, or in the case of some ideal systems of geometry more dimensions. Polygonal systems of objects arranged [67] in definite orders, will probably become much more numerous in the science of the future, and much more variously useful, than they have yet been found to be. But, however complex an ordered system of interrelated objects may become, it will always be developed from a logically reducible fundamental concept of a logical series of objects, a concept whose most general characters we stated at the last time. All exact science, so long as human thought retains its present essential characters, will forever be a variation on the fundamental theme of a one-dimensional ordered system. Whatever the objects we conceive, whether in the moral or in the aesthetic or in the physical world, in so far as we precisely define their relations to systems. But our simple series will always have the fundamental characters of the one-dimensional ordered series.

In recent discussion a name has appeared for any ordered system of real or ideal objects. This is the term “manifold.” A “manifold” is a system that, as we have now seen, is founded upon certain ordered series and results from the complication or interweaving of these ordered series. It may have a determinate number of dimensions. It may lack those manifolds which modern physical science has occasion to deal with in its definition of the various kinds of physical units, - it may be, I say, a manifold definable in terms of a fractional number of dimensions. It may be a manifold that ha the characters of intensive and extensive quantities or magnitudes. It may be a manifold that like any ordered system voluntarily acts as the structure of a whole number series, in so far as every act leads to the next act. It may be a manifold [69] such as a symphony, or a cathedral represents. But whenever we understand the relations present in any such object, we shall find that the system in question is a manifold, having the essential characters of logical order which we have now in general defined. When a manifold is once defined, it will permit us to establish within it for certain purposes, various levels. The significance of conceiving such levels within a manifold we have in general pointed out. Whenever we conceive of levels, we shall define something as equal to something else. And that it is logically significant to do this not merely in the world of quantities but in the moral world we saw in our reference to the various efforts to define men as equal. The logical significance of social equality, in any such case namely one takes a level across certain series, and for a certain purpose either abstracts from those series or declares that they are for this purpose insignificant. But a manifold will in general permit, and will on occasion require, transformations. In fact, a system of transformations may itself constitute a manifold. When we conceive of such transformations in a definite and orderly way, we shall define them in terms of ideal operations. It is the office of mathematical science, as it develops, not only to define various types of manifold, but to define systems of operations especially known to represent in ideal terms all the physical, yes, may I add, all the moral types of transformations that prove to be of importance for the purposes of human thought. There is no reason why symbolic algebra of the future should not learn to represent the most delicate shadings of [69] spiritual relationship in terms of the definitions of manifolds. Such description of life would indeed never be the appreciation of life. The real world can never be defined merely as a manifold. But, on the other hand, in so far as we understand the relations of things we shall learn to define them, whether they be quantitative or qualitative things, whether they be physical or moral things, whether they be masses and energies, or spiritual relationships, in terms of a manifold.

From this point of view one of the tasks of science in its endless comparison of logically constructed manifolds, with empirically given statements of fact, is the effort to construct a manifold inclusive enough to represent in so far as any representation is ideally possible the whole system of the facts present in our experience of all grades. The ideal of an inclusive manifold is that ideal which the modern theory of energy has seemed to begin to realize in a significant way. I do not believe that the modern theory of energy, even in the form which Oatwald has given it, will prove to be capable of defining a manifold whose properties are comparable with the wealth of relations that the physical and spiritual worlds forces upon our attention. On the other hand, I have no doubt that the office of descriptive science is to seek for a system of series, or levels, and of transformations sufficiently complex to constitute a manifold such as represents the whole wealth of human experience. That this ideal descriptive science is at present infinitely far from realization, we all see, but it is something to know that whatever the ideal manifold proves to be, it will so far as we can now see, depend upon an elaboration of the concept of the ordered series. Whatever the [70] world is, it seems to turn in all its constitution upon the concept that ultimately analyzed proves to be the concept of symmetrical and non-symmetrical transitive relations.

But these relations, as we pointed out at the last time, have been defined as dyadic relations. It will be noticed, however, that in a definite and transitive relation we have to mention three objects, “a”, “b” and “c”. I myself think that the definition of transitive and intransitive relations in terms of dyadic relations is an imperfect characterization, and that the logic of the future will prefer to begin defining properties not of pairs of objects but of triads of objects. However, this will not alter the essential induction of our foregoing discussion, which is that the universal forms of scientific conception, whatever else they may do, include at least this constituent which we have defined, the conceptual forms of the series, the level and the system of transformations.

Lecture V.

The foregoing discussions have established the fact that in a very wide range of scientific inquiry certain conceptual forms prove to be applicable despite their relative uniformity and their logical simplicity of type. This closing discussion has to return to the problem: Why are just these conceptual forms so widely applicable? This is a question that, as our opening lecture indicated, must bring [2] us into the presence of very fundamental philosophical problems. We shall have no time to treat such problems exhaustively. I shall confine myself to units. But even these units, I trust, will prove fruitful to many of you.

I.

What we have found is briefly this: Whoever thinks, classifies objects, and attempts to discover the relations that exist amongst objects. Our attention was principally devoted to a study of certain conceptual forms [3] which result from the study of the relations of objects. And in dealing with the relations of objects, we confined our attention to the simplest type of relations, namely to the so-called dyadic relation, -- to those, namely, which exist between the members of a pair of objects.

Dyadic relations, as we further found, may be classified in two ways, viz., first: into symmetrical and non-symmetrical relations; and, secondly, into transitive and intransitive relations. Intransitive relations, although they have much importance in certain regions of science and of practical life, [4] we left out of account. The transitive relations, however, we especially studied. They proved to be of the two important sub-types, which were determined by our other fashion of classifying dyadic relations. That is, certain transitive relations we found to be symmetrical, certain others to be non-symmetrical. Of the transitive symmetrical relations, the relation of equality is one classic type. Another such relation is that of coexistence, in case you regard any object as coexistent with itself.

When, with these relations [5] in mind we turned to consider aggregates or classes of objects, we observed that we have, in the most various sciences, <u>two</u> very notable types of classes of objects defined and used.

Of these two types the first was described as follows: I may have a class of objects such that if I choose at random any pair of objects from that class, and compare them together, I find that there exists a certain fixed and constant relation R, such that one of the members of the pair in question, -- let me call it i, stands to the other member of the same pair, j, in this relation R while this constant relation R is itself a transitive, and always unsymmetrical relation. [6] For instance, there may be a class of objects whose members are all quantities of the same kind, and it may be the case that if I choose any pair of the members of this class, say i and j, then it always holds true that "i is greater than j." Or my chosen class may be a class consisting of men. And the law may hold that if I choose any two men from this class, then always one of the two, say i, precedes j.

Now any class which is subject to such a law constitutes, as we saw, a single one dimensional Series of objects. Thus the quantities in [7] question would constitute a single series of quantities arranged in the order of their magnitude. The men in question, in my other example, would constitute a single row of men arranged in some order of precedence; and so on in case of other such classes. Any such a class constitutes then a Series.

The second type of classes is subject to another law but one closely correspondent to the foregoing one be again, a class of objects. Suppose that in case I choose any two objects, say b and c, from this class, there exists a transitive but universally symmetrical relation S, such that b is an S of c. Then a class of [8] objects of this type constitutes what we have called a Level.

For instance, let the class in question be again a class of opportunities of the same kind. Let it be the law that any pair of the objects that belong to this class are <u>equal</u>. Then all of the quantities that belong to this class stand on the same Level. Any other symmetrical and transitive relation besides that of equality would serve to define a Level.

We thus defined two conceptual forms, viz., the Series and the Level. Our lectures have been largely devoted to preserving how potent and how widely applicable are these two conceptual forms. Their fundamental logical structure is extremely simple. Their value both for science and for practical [9] life is inestimably vast. Their special forms are endlessly numerous.

It was impossible to illustrate the range of application of these forms, without considering still another conceptual form, viz, that of a Transformation. Transformations, i.e., concepts of types of change, are found in all our thinking processes. We say, however, that classes transformations [sic] interest our thought most when they constitute either Series of Transformations, or Levels of Transformations. Whoever conceives the events of a single day's doings, studies series of transformations. The same is true of the astronomer who follows [10] the path of a comet, of the embryologist who studies the growth of an organism, and so on indefinitely in science, whenever a crooked sequence of events or of stages, is considered. On the other hand, our interest in a set of transformations may be in considering what feature or objects or systems of relations, remain invariant through all these transformations; as is the case when, for instance, the quantum of energy present in a physical system remains invariant through all the changes that occur in this system; or as is the case when, in chemical reactions, the mass of the changing matter remains invariant through all these chemical alterations. [11] In any such instance, where our interest in a set of transformations lies in taking notes of what they leave unchanged, the transformations in question are conceived as lying, in some respect, upon the same level. Other instances of levels of transformations could be found wherein our interest is in the fact that these transformations lead us from one series to a definitely corresponding past or stage of another series, even although we are not taking account of the fact that some such quantum as that of the mass or of energy is remaining [12] same. Thus, whenever any household changes cooks, there are certain directions that have to be given to the newcomer, certain doubts that the housewife feels about the new help, certain well known embarrassments of the housekeeping mechanism which accompany so momentous a change; and in this sense all the transformations known, as changing cooks lie upon one level; because they all occupy corresponding places in the various series of occurrences which have to do with adjusting a household to the service of a new cook. [13]

Series and Levels of transformations, then, are very significant both in science and in practical life, and when we consider how vast is their number, and how numerous are their special forms, we are able to recall a fresh the significance of the thesis with which the foregoing lecture closed, viz., the thesis that, whatever else characterizes exact sciences, the two fundamental conceptual forms, -- or as we may now venture to call them the two Categories, viz., the Series and the Level, are universally present wherever exact science is present, and aid to make such science possible; while, in less exact [14] cases of thinking, as for instance, in practical life, the inexactness of the series and of the levels which can there be defined, much of furnishes the reason for the limitations of the sort of thinking which is there possible.

II.

So much then for the principal Categories, or widely applicable conceptual forms, which our empirical comparison of the work of various sciences has brought to our notice. And now for the question, Why are these Categories so widely applicable?

It is easy, without going very deep into the problems of pure philosophy to make plausible either one of two theses [15] [16] regarding what it is that makes these two categories of the series and the level so widely applicable in such various regions of our experience. Notice that I do not call these two theses supposed theses. But they are at any rate apparently different theses.

The first thesis runs: ‘These Categories of the Series and the Level are so widely applicable because the real world as it exists apart from our human thought actually contains series and levels, which our thought in the first instances, simply finds there.” For this real world, as one may first insist, is a world in space and in time. Now time furnishes to us, everywhere, ordered series of events and does so whether our thought chooses to think in terms or not. The unsymmetrical and transitive relation defined by asserting that one event is the “successor in time” of another event binds any two temporal events together, unless the two [17] are contemporaneous. Hence the time series is a sort of pattern and prototype of all series, for all reality is known to us as involuntary temporal series of events. On the other hand, space, as in

Herbert Spencer's phrase, the "abstract of relations of coexistence," is that aspect of the real world which forces upon our attention the symmetrical and transitive relation expressed by the phrase "coexistent with" or contemporaneous with. Hence space, which again is an universal aspect of the real world, is the prototype of all the levels. So far as all parts of space, and all things or events, that, at one any [sic] time are together in space, coexist, they are all on one level. Moreover, as one may continue, space furnishes to us still other instances of levels. For, in that property of space which permits [18] the consequence of lines and so the equality of lengths to be tested by superposition, we have the classic experience in terms of which we exemplify what we mean by quantitative equality. All physical measurements reduce in the last analysis, to measurements of lengths, equal intervals of time can be exactly defined only by presupposing the conception of equal intervals of distance in space, -- a fact of which a glance at your watch face may at any moment remind you. And even so, the mass of matter is defined in terms of the concept of acceleration, which depends on that of length. But what you mean by equal lengths measured upon lines, you may and do discover, so one may insist, by actually trying experiments with foot-rules, yard-sticks, tapes, dividers, and other more or less exactly definable objects. Thus the two classic instances of symmetrical [19] transitive relations, namely of the relations of coexistence and of equality, are furnished to you by certain of your experiences of the properties of space. As for what we have called "levels of transformations," the space-world also presents them to you in their most familiar form, whenever you observe that any solid body may be moved about freely in space without altering any of its dimensions. Thus then, the so-called "axiom of free mobility", which asserts that movements in space do not as mere movements alter the form of the objects that are found in space, expresses the most familiar generalization that experience furnishes to us regarding [20] a set of transformations such that they leave certain real things or aspects of things invariant. All the transformations of such a set lie upon one level.

In brief then, just as the time-series is the prototype of all series, so the space-world is a sort of primal <u>locus</u> of levels. To be sure, we do possess the concept of contemporaneous events, even apart from our concept of space, as the coexistence of feelings and of sensations in the same psychological instant exemplifies.

So that time seems to permit levels. But this instance of purely temporal levels derives most of its interest to science from its relation to the facts of the world in space. And still more obviously, space comprises within itself countless serial orders of [21] objects, -- the points on a line, -- the successive sections of a solid by a series of planes, the inestimably numerous series of physical objects in the real world; and so on indefinitely. But the serial order of things in space, it may be insisted, seems to be a secondary result of their existence; because we are able to run over the various parts of space in successive series of acts of our own while serial order in time is the most essential characteristic which holds true of events, viewed just as events.

In any case, however, as one may thus insist, the real world in space and in time, so soon as we enter it, appears to us to be a realm of series and of levels. The prevalence of these categories in our thought [22] is then at least in great part a <u>result</u> of their prevalence in the temporal and spatial order of the real world.

Yet not even thus do we exhaust the sense in which the real world appears to be the realm where our two categories are found to have been at home, even before our human thought ever set out upon its quest for order. For these two categories of the series and the level have a still deeper relation to the innermost structure of reality, so far as reality is in anyway accessible to our science. This namely, is what the modern doctrine of energy shows us. For the changes of the energy of any system follow a certain determinate series, according to the law [22a] that such quantities as the temperature of the various parts of a physical system tend towards equality, through the transfer of heat from the hotter to the colder parts of the system; while, as we also know, water runs down hill, electrical energy tends towards a distribution that abolishes differences of potential, and so on. Thus, in general, the energies of the world tend to a certain level of distribution, -- a level never actually attained within the range of our experience but always definable with reference to certain serial orders of unstable distributions of energy; while the events of the physical world form series such as always approach the ideal level of distribution. [23]

Thus the events in the physical world do not merely form series, but also form series which have determinate relations to certain levels. The familiar fact of the existence of the sea-level as an approximately constant fact of terrestrial nature, -- a fact which the tides and the winds and the currents endlessly disturb, but which varies only within certain relatively narrow limits, -- this fact is itself but a special instance of the way in which the real world forces upon our attention the existence of certain approximate levels of the distribution of certain forms of energy, as well as the existence of a tendency of energy to reassume such distributions when they have been disturbed. [24]

Yet it is not only the theory of energy that thus exemplifies the importance in the real world of the categories of the series and the level. The series upon which the modern doctrine of evolution insists, are real, and they are characteristic of large regions both of the inorganic and of the organic worlds. If the doctrine of evolution is in any form sound, then there are real series, of organic forms and processes whose order determines every detail of every living organism. Your own organic characters, for instance, are what they now are because of the whole series of your ancestors, back to the remotest times. [25]

And that mingling of coexistent characters, derived from various amounts, which constitutes your organic temperament is an instance of a level, due to the [correlation? *image blurry*] of many series, or again, the geographical distribution of the organic forms, that are at any time present on the earth's surface, constitutes a level that nature draws across countless such evolutionary series such as determine the sequence of organic forms.

Why, however, should I multiply these illustrations of the real significance of our categories. Every science furnishes its own examples. Mendelief's arrangement of the chemical elements is a combination of series and levels; and it stands for a system of natural facts. The law of supply and demand in economics is a statement of a tendency towards a certain level and of a tendency which also determines certain series of economic events. [26]

III.

And yet, as I must insist, there is another thesis regarding our two categories which it is certainly necessary for any one to consider who has once learned the lesson that Kant's analysis of human knowledge has taught us. This thesis is that the prevalence of these two categories, the Series and the Level, is due, in our science, to the nature of the human intelligence, and not to anything that can be understood apart from the consideration of this [illegible] the human intelligence. Whether this thesis is really opposed to the preceding thesis, namely to the thesis that the real world furnishes the true basis for these categories, [illegible] may remain for the moment, a question. The solution [27] of that question depends, of course, upon what you ultimately mean by talking of the real world at all. But postponing for the moment that issue, let us proceed to consider what evidence there is for saying that the wide applicability of the concepts of series and of levels, in our science, is actually due to the fashions of procedure of the human intelligence in its dealings with phenomena. This thesis at any rate appears to contrast sharply with the foregoing thesis. But let us consider the evidence for it dispassionately.

We spoke, but a moment since, of space and of time as the regions where the concept of the level and that of the series find, respectively, their primary and natural expressions. Space we then found to be the great locus of levels; and time we regard as especially [28] the realm of series. Space as we saw contains also series as well as levels; but that appeared to us to be at least largely due to the fact that we can consider the parts and the elements of space successively. Time permits levels, and this seems to be due to the fact that contemporaneous series of events can in some cases be apprehended by the mind even when we do not refer these events to different regions of space. But such apprehension of coexistence apart from spatial relations, seems to us to be of less importance in our experience of reality.

But now when we review the [29] very facts which we thus summarized, we see that our account of them borrowed its whole meaning from a certain conception which we are accustomed to use when we talk of the real world, namely from the conception that the real world consists of an order of things and of events in space and in time either this conception is or is not well founded, it certainly is no direct expression of what you and I experience from moment to moment. For the space-world that we assumed as the real world, when we defined space as the locus of levels, is not identical with the space world that at any moment we have forced upon our perception. On the contrary, the space-world of [30] the geometer, and of physical science, is known to us mortals as a highly artificial construction of our own intelligence, -- a construction of whose validity we feel sure, but whose reality no direct momentary experience of ours ever can suffice to demonstrate. It seems fair then to say that it is the way in which we ourselves are irresistibly disposed to conceive things which gives what we call space and time their characters.

For consider: the space-world, as you conceive it, contains, countless coexistent objects which you do not now observe. You conceive, for instance, that the portion of this room which is behind your back coexists with the portion that is in front of your eyes. I defy you, however, with your present sense of sight, to observe at any one moment, or through anyone act of apprehension, [31] that the whole of what you thus conceive to be true about even the visible parts of this room is a visible fact or is apprehensible in terms of any other direct sensory experience. Turn your head to see whether the wall behind you is observable and the blackboard in front of you will disappear from your view. Undertake to verify the fact that your left hand neighbor is a being visibly coexistent with yourself, and your right hand neighbor at once tends to cease to be visible. It is precisely as impossible to see in any one act that all the visible things in this room actually coexist, as it is to see your own ears. And the reason for this failure to observe a coexistence which you all the while are irresistibly impelled [32] to conceive, and of course to regard as real, is, in case of your two ears, and of the other visible objects not now directly seen by you, very much the same. The coexistent objects present in this room are objects of a possible attention on your part. This possible attention you can direct now to this, and now to that object. But, the field of your attention is notoriously and extremely narrow. Only within that narrow range can you ever verify that coexistence which you conceive to be characteristic of your space-world. If however you ask why you so potently and powerfully believe that this coexistence [33] which you can verify only for so narrow range of objects, actually holds true of the indefinitely vast range of objects that you conceive to be present in the space of the real world; then the answer is that you conceive this to be true because certain very deep needs of your thought seem to you to require the assumption of the truth of this conception. You need namely, at every moment, to conceive of all the material objects in this room as in certain respects upon the same level of coexistence. As a fact you can only see first what is in front of you, and then, turning the head, what is behind you. [34] But in your conception, for reasons which I have no time at present to analyze exhaustively you need to conceive of what is in front of you as coexistent with what is behind you. Hence you conceive as upon one level of coexistence what experience presents to you only serially.

From this point of view, it would appear that the world of what is sometimes called pure experience presents to you rather series of various sorts than those levels of coexistence which constitute the world of physical facts as the latter coexist in space. And your belief that the facts do coexist in space would appear to be due to certain needs [35] of your own intelligence, -- needs which guide your interpretation of the realm of phenomena. And so far the concepts of the level, to say the least, would appear to be forced upon you, in this respect at all events, because of your character as the intelligent interpreter of your experience. This however is what the present thesis asserts of the categories now in question.

But to the considerations thus advanced there is, however, a familiar answer which to many minds does indeed seem once more to indicate for the conception of the levels of coexistence a character relatively independent of your own thoughtful way of interpreting your experience. This answer is as follows: Objects are [36] on a level when, as we have said, any two of these objects stand in a certain constant relation which is both symmetrical and transitive. Now it is true that I cannot now see that the objects in front of me and the objects behind me visibly coexist in the same field of vision. But I can indirectly establish through my successive acts of perception, a transitive symmetrical relation between these objects; and I can do so thus:--

I can first look at one of them, then at the other, and then back again. Hereupon I find that the objects are such that I can pass from either of them to the other, so that I can set them in either sequence in my visual experience and this, it will be insisted is a fact that is not dependent upon my will. For unless [37] these objects are such as are usually said really coexist in space, I in general cannot do this. Thus if, to use Kant's famous instance of Kant's, I were observing a ship that was actually moving down stream with the current, I could first see the ship higher up the stream, and then lower down the stream; but I could not in this case reverse the order of my perceptions, and see it first lower down the stream and then higher up. In the case of the floating ship I should accordingly be observing what is called a real succession. For the relations presented to my experience from without would then be essentially unsymmetrical. I should be dealing with a real series, and not with a level. In case however, of the [38] coexistent parts of a whole present in visible space, I have indeed primarily an experience of serial succession; but this succession, in just this case of the coexistent parts of the whole, I find to be essentially reversible. If a is one visible object present in a given space, and b is another, I may find indeed that for me, a and b never visibly coexist at any one instant for my sight. But, I do also find that in my experience, according as I look this way or that, the sight of a may first precede the sight of b and may then follow the sight of b, so that, for me, the empirically given relation between a and b is expressed by saying that "In my order of perceptions a is at pleasure either the predecessor of b or the successor [39] of b, or is both the predecessor and the successor of b. Now this relation, which, whether I will it or no, I find to be the true relation of a and b, (e.g., of the parts of this room) is indeed a symmetrical relation. For although the relation "successor of" is unsymmetrical, the relation "predecessor and successor of", or the relation "either predecessor or successor of, indifferently" is a symmetrical relation. Now, as one may hereupon insist, it is just this which experience actually shows us to be the relation between any two visible things which coexist in space.

Thus, once more, as it may be asserted, the levels of coexistence are vindicated as facts which the objective [40] order of our experience forces upon us, and which accordingly are not due to the mere needs of our interpreting intelligence.

Now it will be noted, in any case, that this particular method of vindicating for the levels of spatial coexistence a character that in certain respects is not dependent upon the fashions in which our human intelligence works in our interpretation of facts, is a method which ascribes to these levels of coexistence a peculiar and rather unexpected character. We had supposed, while we were talking of them [41] as well-known aspects of the real world, that the symmetrical relations which made them real were direct and comparatively simple facts of experience. We now see that these levels are given to us at best only in an indirect way; and that at least, in the case just now in question, the symmetrical relation which makes the level real is given to us in the form of a complex of two unsymmetrical relations, each of which is the converse of the other. To say "a coexists with b" now means "a may be made, in my experience, to follow b or to precede b, indifferently". From this point of view, unsymmetrical relations and consequent series, would [42] be, for us, the primal facts of our experience. Levels would result, not in general from the directly observed correlation of numerous series, but from the discovery that certain series of facts in our experience are reversible. The narrowness of our consciousness forbids us to correlate many facts at once. But, from this point of view, we can and do experience many series of facts. We discover the reversibility of some of these series. But if a given series is characterized by some unsymmetrical relation R, the reverse of this series gives us a second series containing the same terms, but characterized by the converse of the relation R. Call this new relation “R-converse”. Then what [43] experience shows us, in case we discover a level, would be, in general, that any two of the objects that we are considering belong both to the R and to the R converse series, and so, are such that either one of them, say i, is “both R and R converse of” of the other, say of: The relation “both R and R converse of”, is however a symmetrical relation!

Do we actually discover the existence of levels in this way? I have no time to devote to this question any adequate consideration. But any student of the modern psychological theories of our consciousness of space knows how much evidence there [44] is in favor of this general proposition, so far as the various levels of our space consciousness are concerned. I am not disposed, however, to generalize at this point hastily. Our general argument has, at the moment, only this interest in the issue. We are considering whether our assurance that the real world contains series and levels is due to characters which our experience of the real world forces upon us, apart from the special interests and modes of interpretation which characterize our own intelligence. In defense of the thesis that the usefulness of the concept of series and of levels is due on the whole [45] to the needs of our own interpreting intelligence, we pointed out that the unity of the world in space,- the classic instance of a level of coexistence, is an unity that we never find directly given, but do nevertheless conceive as real. And we suggested that this whole concept was apparently due to the internal needs of our own conceptual process, since to believe as we do in the unity of the world in space involves countless beliefs about the real world which no direct experience of ours ever verifies, but which our sense of the reasonableness of our convictions somehow seems to require. In brief, we need, it was suggested, to conceive certain facts as on a level. Hence we do so.

But to this contention the reply is made that although experience does not present to us at once the unity of the real world in space, experience does [46] indirectly teach us that the things in space have symmetrical and transitive relations. And so it was further asserted, experience does this by presenting to us certain reversible series of experiences. It is, in brief, by giving back and forth between places that we learn of their coexistence. It is because of the reversibility of certain of our movements, and of their results, that we learn of the properties of space. And the suggestion is made that perhaps all our consciousness of levels (in so far as these exist on a scale too vast to be grasped at once within the narrow unity of our consciousness) is due to our experience of reversible series, and to our experience of the contrast between these and the irreversible series which we also find in our experience. [47]

Well, let us for the moment admit this entire conception, and so admit that our assurance of the reality of such levels as are too vast and complex for us directly to observe, is reducible to our assurance that certain series possess a reversible character. Then the problem is to what makes the concept of the series and of the level so potent reduces itself, provisionally, to the single question: What makes the concept of the series so potent?

IV.

Here however, at length, we meet with a decisive issue. The prototype, at least in the realm of our ordinary experience, of all empirical series, as we observed in the earlier part of our statement, is the time-series. [48] Now what assures us of the reality of the time-series. In the form of time-series, all series, reversible and irreversible, just in so far as they are observable by us at all, must come to our empirical consciousness. In the temporal succession of our own experience, we must search through the various regions of space, must test equalities of all kinds, must observe dependencies of all types, must take account of all transformations that experience furnishes to us, must in brief acquire the materials upon the basis of which all our notions of the real world are formed. Thus whatever be the logical nature of serial order, our experience of serial order has to be a temporal experience. But, when do we experience the reality of the sequence of events in time. Now? Yes, if you mean by the <u>now</u> [49] experienced sequence the brief span of the contents of the present moment. But we certainly do not now experience what happened during the nineteenth century. We now conceive that certain events then happened. We do not now perceive yesterday's events, nor yet tomorrow's. Our conception of those events is obviously an interpretation of certain given memories, images, names, suggestions, interests, hopes, demands. The present time does indeed involve an experience of a relation between the earlier and the later content of our present passing consciousness. We conceive this relation as extending indefinitely into both past and future, and as being universally both unsymmetrical and transitive. That is, a character of this relation which our present consciousness suggests, we universalize, we define as holding for all earlier and [50] later temporal events, and so we conceive that all which ever has happened to us, or which ever will happen to us forms a single series of experiences. In other words, we constantly conceive of that unity of a single experience of which Kant so well tells us. It is only by virtue of conceiving this moment as a stage in the united process of a single experience that we got any possession at all of the concept. Of the real series of temporal events, just so far as that series extends beyond the narrow [illegible] series of the present moment.

In vain then, do you point out that we believe in the coexistence of the things of the real world because experience constantly forces upon us the fact, given apart from our con-[51] ceptual processes, that some of the series of events that occur in time are reversible, while some are not. It is in vain, I say, that you do this, in case you thereby seek to relieve our whole system of human experience of its character of being a system that [illegible] we accept through and through [illegible] because the needs of our intelligent process of thoughtfully interpreting what is now given to us, force us to define our experience as a system processing [illegible] and to accept [illegible] because it comes to us merely from without, as a brute fact of an external order, but rather, because we are always engaged in interpreting the data of present experience from within the fold of our own idealizing activity. The fact is that we believe the real world to contain series, because without conceiving [52] series of facts we cannot make our present life as thinkers and as active beings intelligible to ourselves.

At any instant namely, we have something given to us, and we need to do something with this which is given. That is, the present datum means something to us, implies something, leads over to a deed of some sort, arouses a response, sets us at the business of idealizing its contents. And we proceed to idealize these contents by giving them a place in a system; and so any present datum can get a place in our attention only in case it somehow cooperates in our business of defining our own purpose as thinkers who conceive our world as a system. Now it happens that when we undertake to define these purposes of our thought. We do so by conceiving of series of [53] activities, and of data correlated to these activities. These conceived series of data and of activities, are the stuff out of which we weave our whole conception of reality. And the conceived reality therefore contains for us series, because we need to conceive that it does so.

So then over against the thesis that the real world in space and in time everywhere shows us facts serially arranged, we may and must set the equally verifiable thesis that the real world in space and time is never given to us as a directly experienced datum at any moment of our lives, but is always [54] conceived as a system of possible experience, or of experience not now our own; while this system we always now conceive in accordance with the present requirements of our thoughtful activity. One of these present requirements, however, involves the conception of series of past and future experience, and of endlessly numerous ranges of possible experience. Hence we conceive the real world as a system of serial orders of data. We conceive some of these serial orders as essentially (54a) reversible, and some as not reversible. The reversible series in their totality constitute that realm of actual or of possible experience which we most commonly conceive as the world in space. The irreversible series especially characterize the world of time-sequence. But that we thus at any moment conceive our world, or that in any other way we conceive series and levels as present in it, is due, at each instant, to the thoughtful interests which then and there determine the fashions of our conceptual activity. [55] What applies to the spatial and temporal order in general, applies equally well to the special ranges of serial orders, and of levels, such as the series and levels involved in the doctrine of evolution, or those involved in Mendelief's arrangement of the chemical elements. All such objects are to any human thinker, when he thinks, not in their wholeness, presented contents of his experience of reality. They are conceived objects, which he conceives thus because by conceiving them thus he gives to the conceived system of our human experience [56] systematic order and wholeness. All such objects then express our human conceptual needs, as we define those needs at any moment of our experience.

V.

So much in general for the thesis that the use of the conceptions of series and the level in our sciences is due to the needs of our thought in its undertaking to conceive our experience as a system.

In fact, how can you undertake to conceive your present experience as a stage or as a transitional phase [57] in a system of experience. Only, of course, by conceiving, as you always do, that your present experience is a sign or symbol of contents of experience not now present to you. You see the headings in the newspapers. They are signs to you, either of events in the war in the far east, or else of intentions on the part of newspaper correspondents that you should believe thus or thus about the war in the east. You see a star in Perseus, glowing for a few nights and then fading. That light becomes a sign to you [58] of far off cosmic events, i.e. of contents that might have been experienced by you had you been, as observer otherwise situated in the universe.