engage permanent truth from error in current doctrines of the relation of the two.

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REVIEWS AND ABSTRACTS OF LITERATURE

The Relation of the Principles of Logic to the Foundations of Geometry. J. ROYCE. *Transactions of the American Mathematical Society,* July, 1905. Pp. 353–415.

The subject of this essay is not so broad as a glance at its title might suggest. For, first, by the 'principles of logic' are meant simply the formal rules of an algebra of logic, the canons governing the manipulation of assumed concepts of absolute distinctness and fixity. In other words, it is the logic of the 'exact sciences' with which we have to deal, understood as a deduction from a definite set of ultimate (and, therefore, mutually independent and indifferent) premises. Secondly, by the 'foundations of geometry' are to be understood such a set of premises, selected with a view to their sufficiency as a source for the derivation of geometrical truths. Finally, by the 'relation' between these 'principles' and 'foundations' is meant a remarkable similarity, the extent and significance of which it is the chief object of the essay to determine.

The general nature of this similarity may be briefly explained as follows: The terms of the fundamental propositions, upon which an exact science (according to the above-mentioned interpretation) is based, can have no meaning other than that which their place in those propositions gives them. The words used to denote these terms are, therefore, in themselves absolutely meaningless, and no loss is suffered when they are replaced by algebraic symbols. The only relation which is regarded as ultimately subsisting between such terms is that of copresence in determinate groups; and the fundamental principles of the science are simply postulates establishing the existence of these groups and exhibiting the conditions under which the elements of one group may enter into other groups. Now when the postulates of logic and of geometry are stated in this symbolic fashion, it is possible to present them in a form in which they are almost entirely identical.

Professor Royce's paper is largely a restatement of a theory advanced by Mr. A. B. Kempe in an essay of similar scope, in the *Proceedings of the London Mathematical Society* for 1890.[1] In this essay Mr. Kempe sets forth a development of the algebra of logic, using as the fundamental relation between classes not (as ordinarily) that of inclusion, but a peculiar sort of 'between' relation. This relation, as defined in ordi-

[1] The reviewer regrets that prolonged illness has prevented his reading Mr. Kempe's essay, for knowledge of which he is dependent upon Professor Royce's account.

nary terms, is that which one class bears to two others when it includes their common extent and is included within their total extent. Mr. Kempe, however, does not so define it, but, on the contrary, regards inclusion as a mere special case of the between-relation; *i. e.,* the included class simply lies between the including class and a particular class called zero. The between-relation is itself defined without reference to the idea of inclusion at all, solely by means of a set of symbolic postulates. On the basis of these postulates, the whole algebra of logic is readily developed—but with one important peculiarity. No means are provided for distinguishing the zero class and the universe class from any other similarly related pair of classes. They must, therefore, be regarded as *arbitrarily* fixed upon. This, however, does not appear to Mr. Kempe (or to Professor Royce) as a defect in the theory. It is regarded as a decided advantage, indicating the superior generality of the discussion to that based upon the relation of inclusion.

It is well known that in the modern logic of geometry the between-relation of points in a straight line occupies a place of fundamental importance comparable to that which the between-relation of classes occupies in Mr. Kempe's logic. In many respects these two relations are remarkably similar. A surprisingly long list of elementary properties can be given which belong equally to both. But there are two fundamental differences.

I. If the *points* b and c are both different from a and d[2] and lie between them, neither a nor d can lie between b and c. This does not hold for logical classes; and furthermore,—

II. For every *class* b between a and d, another class c exists between a and d, such that both a and d lie between b and c.[3] Such classes as b and c will be referred to below as 'conjugate mediators.'

There are various other differences between the logical and the spatial order, but all more or less closely connected with these two. The most remarkable is the existence of logical 'negatives' or 'obverses,' which (like opposite points in a spherical surface) have every other class in the universe between them.

From reasons such as these, Mr. Kempe, followed by Professor Royce, concludes that the system of logical classes may be regarded 'as much *more general and inclusive* than the system of the points of space.' That is to say, "One may view the points of a space as a select set of logical elements, chosen, for instance, from a given 'universe of discourse.'" This thought Professor Royce recognizes as the essential conception at the basis of Mr. Kempe's discussion, and it is equally essential to his own. The importance of the thought he explains as follows: "The relations amongst logical entities are, in any case, the most fundamental relations that we know. Experience shows us in the outer world those

[2] In Mr. Kempe's usage, every element lies 'between' itself and every other. For brevity's sake we shall hereafter ignore this fact.

[3] To verify this law, substitute $m + n + o$ for a, $o + p + q$ for d, and $n + o + p$ for b,—a perfectly general supposition. Then $c = m + o + q$.

ordinal space relations which geometry generalizes in the concept of 'between.' But our own thinking processes show us the meaning of the logical relation [of inclusion]. The latter relation, then, is more suited to be the basis for a theory of the logic of an exact science, in case we can only so define and restrict its application[4] that our ideal geometrical relations can come to be viewed as *special instances* of those forms which we can develop by the use of pure logic" (p. 355; italics mine).

If space permitted, we might be pardoned for stopping to question Professor Royce's antithesis of 'experience,' on the one hand, and 'our own thinking processes,' on the other. This inquiry is rendered unnecessary, however, by the fact that Mr. Kempe's essential thought involves a very serious confusion, due apparently to his not having observed the very different logical significance of the two differences between the logical and the spatial order which we numbered I. and II. above. The confusion in question is that of the *inclusiveness* of a system with its *generality*. The proposition contained in I. does indeed place a specific limitation upon the system of points in space, from which the system of logical classes is free; and the latter system is in so far the more general. But it is to be noted, that the mere freedom from this limitation is by no means equivalent to the proposition of II.; just as the denial of the application of this latter proposition to the points of space is by no means equivalent to I. In other words, II. imposes as veritable a *specific limitation* upon the system of classes as I. imposes upon the system of points. This fact seems to have been concealed by the circumstance, that whereas I. is clearly a restriction upon the otherwise possible multitude of points, II. apparently (though only apparently) enlarges the possible multitude of classes. The fact remains, however, that it is as positive a specification of the system of classes, that it shall contain conjugate mediators, as it is of the system of points, that it shall contain nothing of the sort. We can not, therefore, follow Mr. Kempe (and Professor Royce) in the opinion, that 'our ideal geometrical relations can come to be viewed as *special instances* of those forms which we can develop by the use of pure logic.' What we have here is two mutually exclusive species of the same genus.

While, however, the logical system is no whit more general than the spatial system, it is truly much more inclusive. That is to say, the spatial system would have to be extraordinarily enlarged[5] if it were to be made completely parallel to the logical system. There is in what we have said nothing at all to oppose the suggestion, that we regard 'our' space (or any other space) as a selection from a space thus ideally enlarged. But, for the reasons above given, such a procedure would appear to be wholly uninstructive.

[4] As he believes Mr. Kempe has done by reducing it to the logical between-relation.

[5] The reader should understand that this 'enlargement,' *i. e.*, the addition of conjugate mediators, means much more than the addition of infinite dimensionality to space.

There are, moreover, other serious difficulties attaching to Mr. Kempe's conception. Our whole knowledge of the multidimensionality of the logical system depends (as is apparent from Professor Royce's treatment in §§ 122 and 123) upon the assumption of conjugate mediators. Hence, when the system of logical classes is reduced (essentially by the elimination of conjugate mediators) to the inferior complexity of the system of points in a space, it remains an open question whether the reduction may not have gone too far. That is to say, there remains no warrant for asserting the multidimensionality of the logical system thus reduced. It may be of infinite dimensions, but we can demonstrate the existence of no more than a single line. Professor Royce and his predecessor apparently assume without question the possibility of selecting from the conjugate pairs in such a way as to leave the infinite dimensionality of the system uncompromised. In the absence of explicit proof it may further be questioned, whether the parallel-axiom is rightly said to concern only 'the limitation of the selection of the lines admitted into a given system' (p. 411). For it might be hard to reconcile the selection of one out of an infinite multitude of otherwise possible parallels with the preservation of the continuity of angular rotation. It may be noted, that such a selection, if applied at any point of a Lobatchewskian plane, would at once reduce it to two mutually perpendicular straight lines.

We have hitherto spoken of Professor Royce's paper only in so far as it is, as its author says, 'a restatement of Kempe's logico-geometrical theory.' It remains to consider the original aspects of the present paper. These belong almost wholly to the logical side of the discussion, the connection with geometry being effected in essentially the same manner as by Mr. Kempe. The principal novelty is a change of starting point. Instead of the between-relation, Professor Royce takes as fundamental another relation, which has the methodological advantage of obtaining among groups of any number of elements greater than one, and also of being absolutely symmetrical. This is the relation in which a group of logical classes stand, when they have no common extent, and together exhaust their universe. Of course, the O-relation (as he calls it) is not defined at the outset in any such way. On the contrary, the notions of inclusion and exhaustion have to be regarded as merely incidental phases of the O-relation itself. Accordingly, the O-relation must first be defined in its own terms, so to speak; that is to say, by a set of wholly symbolic postulates. This task is executed with remarkable ingenuity. By far the greater part of the essay is devoted to a development of the algebra of logic upon this basis. Of especial interest in the course of the discussion is the treatment of 'conjugate resultants,' an extension of the notion of 'conjugate mediators' mentioned above. The logic thus developed has the peculiar character (already noted as belonging to Mr. Kempe's logic) of reducing the zero element and the universe element to the dead level of the other elements of the universe. Logical addition and multiplication, for example, are defined with reference to any element as 'origin'; and it requires the arbitrary selection of a particular element as invariable origin—the zero of ordinary symbolic logic—to reduce these operations to their accustomed form.

Needless to say, the details of this development are worked out with absolute accuracy. A few harmless inelegancies may be noted—as, for example, the fact that the premises of §§ 23 and 24 are oversufficient—but these are quite trivial. Some useless pains are expended upon a distinction between 'equivalence' and 'identity.' As identity means no more than equivalence for all possible purposes, and as equivalence is here defined with réference to the only purpose which the entities in question serve, *viz.*, presence in *O*-collections, the distinction appears to be superfluous.

One fact we may perhaps be pardoned for mentioning, and that is that from beginning to end the essay is exceedingly interesting. The casual reader who opens the essay in the middle and lets his eye pass down an average page, may well be dismayed by the show of abstruse algebraic symbolism. But alarm is needless. The author sets out from the beginning of things, and all the fearful-looking formulas are fully explained as he goes along. Scarcely any technical knowledge of any sort is necessary to comprehend the whole discussion.

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JOURNALS AND NEW BOOKS

REVUE PHILOSOPHIQUE. February, 1906. *Pragmatisme et pragmaticisme* (pp. 121–146): A. LALANDE. – A history and criticism of pragmatism and humanism. The philosophy is a realistic one, a combination and culmination of all nineteenth-century tendencies, with one exception; this single tendency which has not been taken into account is the sociological one. Pragmatists have tried to make truth individual, but their own realism forces upon us the interpretation that truth is objective in the sense that it is the limit approached by the totality of human opinions. Intellectualism is a means to the end of socializing all individual judgments; it is thus pragmatic. *L'ironie* (pp. 147–163): G. PALANTE. – Irony is an individual, not a social, emotion; but its reference is chiefly to society. It bases on a dualism between feeling and reason, or upon the difference between our aims and their results. Essentially pessimistic in tone, it is not an intellectual, but rather an esthetic attitude. *De l'avarice* (Conclusion) (pp. 164–201): ROGUES DE FURSAC. – In the miser the egoistic feelings are unbalanced; positive love of life, as such, is often lacking. Yet this does not imply a suicidal tendency, for the miser lives for the love of wealth; having low sensitivity, he never finds life positively intolerable. Lacking in all social feelings, he is vain, envious and suspicious; a mystic, too, losing himself in ecstasies over abstract wealth. He belongs to the very opposite class to the hypnotic, being neither impulsive nor influenced by suggestion. He is rarely criminal,