Types of Order and the System Σ

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TYPES OF ORDER AND THE SYSTEM Σ.

IT is a commonplace of current theory that mathematics and exact science in general is capable of being viewed quite apart from any concrete subject matter or any system of physical facts to which it may usefully be applied. Geometry need not appeal to any intuition of spacial complexes or to a supposititious space form; it has no need to rely upon diagrams or make use of 'constructions.' Arithmetic makes no necessary reference to the sensible character of collections of marbles or of areas. Dynamics does not require the dubious assumption that the 'moving particles' of which it treats are possible of experience or verifiable physical entities. The 'points' of geometry and kinematics, the 'numbers' of arithmetic, and so on, are simply terms,—x's, y's, z's, entities, anything,—and the question what concrete things may be successfully regarded as such x's and y's is a question of application of the science, not one which need be considered while the system itself is in process of development.

If considerations of usefulness and of application are important in determining what assumptions shall be made or what systems developed, still such pragmatic considerations are principles of selection amongst actual and possible systems, and not internal to the systems themselves.

An arithmetic, a geometry, a kinematics, is thus capable of being viewed simply as a complex of relations and operations (relations of relations) which obtain amongst entities the nature of which, apart from those properties which follow from the relations assumed, is wholly indifferent. Such a system may, in fact, admit of various interpretations and applications, more or less useful, all of which satisfy the requirement that these relations and operations be valid. As Professor Royce is accustomed to put it: a system of science is a type of order, the distinguishing characteristics of which are the kind of relations—symmetrical or unsymmetrical, transitive or intransitive, etc.,—which obtain among its terms, and the relations of these relations, by means of which the terms are 'ordered' and the relations 'transformed.'[1]

The growing recognition of the advantages of so viewing systems of pure science is one of the prime motives for the present interest in symbolic logic, or logistic. For logistic is the science which treats of types of order. One may reach the particular type of order which it is desired to portray—the arithmetic or geometry—by further specification of that minimum order which must obtain among entities if they are to 'belong together' in a set or system—the order of logic. This can be done in a variety of ways, which may be roughly divided into two groups. These two methods are distinguished by the fact that in the one case the 'numbers' of arithmetic or 'points' of geometry are treated as (conceptual) complexes having a definite internal structure, while in the other the 'numbers' or 'points' are the simple and indifferent terms, the x's and y's of the system. The former mode of procedure is best illustrated by the investigations of Russell's *Principles of Mathematics* and *Principia Mathematica* of Russell and Whitehead. The other method is exemplified by Dedekind's *Was sind und was sollen die Zahlen*, by the *Ausdehnungslehre* of Grassmann, and by the paper of Mr. A. B. Kempe, "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points."[2] But this second method appears in its best and clearest form in the paper of Professor Royce on "The Relation of the Principles of Logic to the Foundations of Geometry."[3] Each of these procedures has its advantages and its difficulties. Of late, the first method has received a disproportionate share of attention. For this reason, if for no other, I deem it important to call attention to the second method in general and to Professor Royce's paper—its notable exemplification—in particular.

Professor Royce generalizes upon certain relations previously

[1] I do not know that Professor Royce has anywhere printed just this statement, and my way of putting it may not be satisfactory to him, but Harvard students in "Philosophy 15" will remember some such formulation.

[2] *Proc. London Math. Soc.*, Vol. 21, p. 147. See also his earlier "Memoir on the Theory of Mathematical Form," *Phil. Trans.*, Vol. CLXXVII, p. 1, and the Note thereon, *Proc. Royal Soc.*, Vol. XLII, p. 193.

[3] *Trans. Am. Math. Soc.*, Vol. 6, p. 353.

pointed out by Kempe, in the paper mentioned above,—certain relations which are fundamental both for logic and for geometry. If $ac \cdot b$ represent a triadic relation in which a and c are the 'even' members and b is the 'odd' member, $ac \cdot b$ is capable of various significant interpretations. If a, b, and c represent areas, $ac \cdot b$ may be taken to symbolize the fact that b includes whatever area is common to a and c, and is itself included in that area which comprises what is either a or c (or both). The same relation may be expressed in symbolic logic as

$$ac \prec b \prec (a + c); \quad \text{or;} \quad \bar{a}b\bar{c} + a\bar{b}c = 0.$$

This relation may be so assumed that it has the essential properties of serial order. Taking it in the form just given and presuming the familiar laws of the algebra of logic, if $ac \cdot b$ and $ad \cdot c$, then also $ad \cdot b$ and $bd \cdot c$. Hereupon we may translate $ac \cdot b$ by 'b is between a and c,' and the relation will then have the properties of the points a, b, c, d, in that order. Further, if a be regarded as an origin with reference to which precedence is determined, $ac \cdot b$ may represent 'b precedes c,' and $ad \cdot c$ that 'c precedes d.' Since $ac \cdot b$ and $ad \cdot c$ together give $ad \cdot b$, if 'b precedes c' and 'c precedes d,' then 'b precedes d.' Hence this relation has the essential transitivity of serial order, with the added precision that it retains reference to the origin from which 'precedes' is determined.

Professor Royce points out to his students that the last mentioned property of this relation makes possible an interpretation of it for logical classes in which it becomes more general than the inclusion relation of ordinary syllogistic reasoning. If there should be inhabitants of Mars whose logical sense coincided with our own—so that any conclusion which we regarded as valid would seem valid to them, and *vice versa*—but whose psychology was somewhat different from ours, these Martians might prefer to remark that "b is 'between' a and c," rather than to note that "all a is b and all b is c." These Martians might then carry on successfully all their reasoning in terms of this triadic 'between' relation. For $ac \cdot b$ meaning $\bar{a}b\bar{c} + a\bar{b}c = 0$ is a general relation which, in the special case where a is the "null" class contained in every class, becomes the familiar "b is included in c" or "all b is c." By virtue of the transitivity pointed out above, $oc \cdot b$ and $od \cdot c$ together give $od \cdot b$, which is the syllogism in *Barbara*, 'If all b is c and all c is d, then all b is d.' Hence these Martians would possess a mode of reasoning more comprehensive than our own and including our own as a special case.

The triadic relation of Kempe is, then, a very powerful one, and capable of representing the most fundamental relations not only in logic but in all those departments of our systematic thinking where unsymmetrical transitive (serial) relations are important.[1] In terms of these triads, Kempe states the properties of his 'base system,' from whose order the relations of logic and geometry both are to be derived. The 'base system' consists of an infinite number of homogeneous elements, each having an infinite number of equivalents. It is assumed that triads are disposed in this system according to the following laws:[2]

1. If we have $ab \cdot p$ and $cb \cdot q$, r exists such that we have $aq \cdot r$ and $cp \cdot r$.

2. If we have $ab \cdot p$ and $cp \cdot r$, q exists such that we have $aq \cdot r$ and $cb \cdot r$.[3]

3. If we have $ab \cdot c$, and $a = b$, then $c = a = b$.

4. If $a = b$, then we have $ac \cdot b$ and $bc \cdot a$, whatever entity of the system c may be.

To these, Kempe adds a fifth postulate which he calls the 'law of continuity': "No entity is absent from the system which can consistently be present." From these assumptions and various definitions in terms of the triadic relation, Kempe is able to derive the laws of the symbolic logic of classes and the most fundamental properties of geometrical sets of points.

[1] It should be pointed out that the triadic relation is not necessarily unsymmetrical: $ac \cdot b$ and $ab \cdot c$ may both be true. But in that case $b = c$, as may be verified by adding the equations for these two triads. Further, $ab \cdot b$ is always true, for any a and b. Thus the triadic relation represents serial order with the qualification that any term may be regarded as "preceding" itself or as "between" itself and any other.

[2] See Kempe's paper, "On the Relation between, etc.," pp. 148–149.

[3] If the reader will draw the triangle abc and put in the "betweens" as indicated, the geometrical significance of these postulates will be evident. I have changed a little the order of Kempe's terms so that both 1 and 2 will be illustrated by the same triangle.

But there are certain dubious features of Kempe's procedure. As Professor Royce notes, the 'law of continuity' makes postulates 1 and 2 superfluous. And it renders entirely obscure what properties the system may have, beyond those derivable from the other postulates without this. For the negative form of the "law of continuity" makes it impossible to assume the existence of an entity without first investigating *all* the properties of *all the other* entities and collections in the system, where some of these other entities and collections exist only at the instance of the 'law of continuity' itself. Consequently the existence of any entity or set, not explicitly demanded by the other postulates, can be assumed only at the risk of later inconsistency. Also, in spite of the fact that Kempe has assumed an infinity of elements in the base set, there are certain ambiguities and difficulties about the application of his principles to infinite collections.

In Professor Royce's paper, we have no such 'blanket assumption' as the 'law of continuity,' and the relations defined may be extended without difficulty to any finite or infinite set. We have here, in place of a 'base system' and triadic relations, the 'system Σ' and "*O*-collections."

The system Σ consists of simple and homogeneous elements. Collections of these may contain any finite or infinite number of elements; and any element may be repeated any number of times; so that *x* and *x*-repeated may be considered a collection, *x*, *x*-repeated, and *y* a collection, and so on. Greek letters will signify determinate collections in Σ. Collections in Σ are either *O*-collections or *E*-collections. *O*(———) signifies that (———) is an *O*-collection; *E*(———) that (———) is an *E*-collection, *i. e.*, that it is not an *O*-collection. Assuming for the moment the principles of the algebra of logic, $O(pqrs \cdots)$ signifies that $pqrs \cdots + \overline{pqrs} \cdots = 0$. [Both the laws of the algebra of logic and the properties of *O*-collections which render them thus expressible are, of course, derived from the postulates and not assumed in the beginning.] It will be clear that the order of terms in any *O*-collection may be varied at will.

'*x* is equivalent to *y*' means that in every collection in which

x or y occurs the other may be substituted for it and the collection in question still remain an O-collection.

If two elements in Σ, say p and q, are such that $O(pq)$ is true, then p and q are said to be *obverses*, each of the other. Since it will follow from the postulates of the system that all the obverses of a given element are mutually equivalent, and that every element has at least one obverse, a 'unique representative' of the obverses of x may be chosen and symbolized by $\bar{x}$. Pairs of obverses will turn out to have the properties of negatives in logic.

Any q such that $O(\beta q)$ is true, is called a *complement* of β.

Any r such that $O(\beta q)$ and $O(qr)$ are both true is called a *resultant* of β.

The postulates of the system Σ are as follows:[1]

I. If $O(\alpha)$, then $O(\alpha\gamma)$, whatever collection γ may be.

II. If, whatever element b_n of β be considered, $O(\delta b_n)$, and if $O(\beta)$ is also true, then $O(\delta)$.

III. There exists at least one element in Σ.

IV. If an element x of Σ exists, then y exists such that $x \neq y$.

V. Whatever pair (p, q) exists such that $p \neq q$, r exists such that while both $O(rp)$ and $O(rq)$ are false, $O(pqr)$ is true.

VI. If w exists such that $O(\theta w)$, then v also exists such that $O(\theta v)$ and such, too, that whatever element t_n of θ be considered $O(vwt_n)$.

From these assumptions the whole algebra of logic can be derived in such wise that the system Σ has the order of the totality of logical classes. To see this, we must first define the F-relation. If $O(pqrs \cdots)$ to any number of terms, we may represent the same fact by $(F(\bar{p}/qsr \cdots)$, $(F\overline{pr}/qs \cdots)$, $(r/F\overline{pqs} \cdots)$, etc., where the rule for transforming the O-collection into the corresponding F-collections is that we introduce a bar, separating any one or more elements of the O-collection from the remainder, and then replace each of the elements on one (either) side of the bar by its obverse.[2] Since the order of terms in O-collections is indifferent, terms on the same side of the bar in any F-relation

[1] See p. 367 of the paper.

[2] This definition presupposes the proof of the principle that if $O(pqr\ .\ .)$, then also $O(\overline{pqr}\ .\ .)$, as well as the proofs which make possible the notation $\overline{pqr}$, explained above. See pages 367–371 of the paper.

are independent of the particular order in which they are written. Also, it follows immediately from the definition of the relation that $F(pq/\overline{rs})$ and $F(\overline{pq}/rs)$ are equivalent. Where the F-relation holds for three terms, it turns out to be identical with the triadic relation of Kempe, and the Kempean $ac \cdot b$ is thus a special case of the F-relation, namely $F(b/ac)$, or $F(ac/\bar{b})$, or $F(a/\overline{bc})$, or $F(\bar{a}/\bar{b}c)$, or $F(b/ca)$, etc., all of which are equivalent. We may, then, define the "illative" relation,—"b is included in c" where b and c are classes, "b implies c" where b and c are propositions, "b precedes c," where b and c are points or terms in one-dimensional array,—as the special case of any of the above F-relation' in which a is the "zero element," or "null class," or "origin." But these F-relations are equivalent, by definition, to $O(a\bar{b}c)$ and $O(ab\bar{c})$. Hence $b \prec_a c$ may be defined to mean $O(a\bar{b}c)$ and $b \prec c$ to mean that $O(0\bar{b}c)$. Thus in terms of the totally symmetrical O-relation, the unsymmetrical, transitive dyadic relation which characterizes both serial order and syllogistic reasoning can be defined.

As is well known, the entire algebra of logic may be derived from a class K, the idea of negation, and the illative relation, hence also in terms of the system Σ and O-collections. The 'zero element' or 'null class' is any arbitrarily chosen member with reference to which all illative relations are supposed to be specified. Such an element 0 itself bears the illative relation to any other, x, since $F(0x/0)$, or $O(0\bar{0}x)$ holds for any element x. The element 1, the "universe" of the algebra of logic, may then be defined as the negative or obverse of the 0 chosen. In the system Σ, 0 and 1 do not differ from any other pair of obverses, apart from the arbitrary choice of a reference element for illative relations. The logical product of two terms, x and y, is then definable as any P such that $F(0x/P)$, $F(0y/P)$, and $F(xy/P)$. The logical sum of x and y is definable as any S such that $F(1x/S)$, $F(1y/S)$, and $F(xy/S)$. P, so defined, will be such that $P \prec x$ and $P \prec y$, while any w such that $w \prec x$ and $w \prec y$ will be also such that $w \prec P$. For S it will be true that $x \prec S$ and $y \prec S$, and any v such that $x \prec v$ and $y \prec v$ is also such that $S \prec v$. S and P are, in fact, the "lower limit" and "upper limit," with reference to the chosen zero element, of all the F-resultants of x and y, an F-resultant being any z such that $F(xy/z)$. These definitions for the product and sum of two elements may be extended immediately to any number of elements, or any collection β, if we replace x and y by "any element of β, however chosen." The usual laws of the algebra of logic, connecting sums and products, terms and their negatives, and the elements 0 and 1 may then be verified for the system Σ. This order of logical entities is contained in Σ in an infinite variety of ways, since any pair of obverses may be arbitrarily chosen for 1 and 0. F-relations and O-relations, not confined to dyads and triads, *are capable of representing this order in a generalized form.*

There is, moreover, a wealth of order in the system which the algebra of logic, even in terms of any polyadic relation, does not require. It is this difference which renders the system Σ capable of being viewed as a generalized space form.

It follows from postulate V that if $p \neq q$, then there is an element 'between' p and q. The postulate states: Whatever pair (p, q) exists such that $p \neq q$, r also exists such that while both $O(rp)$ and $O(rq)$ are false, $O(pqr)$ is true. $O(pqr)$ or $F(pq/\bar{r})$ gives, by definition of the illative relation, $r -\!\!<_q p$ and $\bar{r} -\!\!<_p q)$ or r is "between" p and q. And $\bar{r}$ must be distinct from p and q both, for otherwise, it follows from the definition of obverses, one of the two $O(\bar{r}p)$ and $O(\bar{r}q)$ will be true. Hence postulate V may be restated in the form: For every pair of distinct elements, there exists an element, distinct from both, between them. It is at once obvious that if the elements be "points," and $p -\!\!<_o q$ mean that p is between o and q, postulate V requires that the order of points in Σ should be dense in every direction (with reference to every pair of points). It is further clear that if we take any pair of distinct points, o and z, and postulate t between them, we shall be required to postulate also r between o and t, v between t and z, and so on. Owing to the transitivity of the illative relation, we are thus required to postulate for every pair (o, z) an infinite number of elements in the order $o -\!\!<_o r -\!\!<_o t -\!\!<_o v -\!\!<_o z$. Such an ordered collection is continuous. We have already seen that it is dense. It remains to see that it satisfies the requirement that every fundamental segment has a limit. Consider two selections from the collection, κ and λ, such that if k is any element of κ, every element j such that $j -\!\!<_o k$ belongs to κ, and every element l, such that for every element k of κ $l -\!\!<_o k$ is false, belongs to λ. There is, then, an element, call it S, such that, for every element k in κ, $k -\!\!<_o S$, and if l is any element such that, for every element k of κ, $k -\!\!<_o l$, then $S -\!\!<_o l$. Such an element S is the 'sum' or 'upper limit' of κ, defined above. Hence every fundamental segment has a limit. Any collection thus characterized by a transitive unsymmetrical relation and continuous order deserves to be called a 'line.' Every pair of distinct elements in Σ determines such a line.

For every pair of distinct points, o and q, there exists p such that $F(oq/p)$ and hence $O(oq\bar{p})$. By the definition of the F-relation, if $O(oq\bar{p})$, then $F(\bar{o}\bar{q}/\bar{p})$. Hence if o and q determine a line, $o \cdots p \cdots q$, there exists also a line, $\bar{o} \cdots \bar{p} \cdots \bar{q}$ or $\bar{q} \cdots \bar{p} \cdots \bar{o}$, in which appear the obverses of all the elements in $o \cdots p \cdots q$. But it also follows from $O(oq\bar{p})$ that $F(o\bar{p}/\bar{q})$, or $q -\!\!<_o \bar{p}$. Thus if $o \cdots l \cdots z$ be any line determined with reference to an "origin" o, the line containing the obverses of the elements of $o \cdots l \cdots z$ may be determined by reference to the same origin. And if two elements of $o \cdots l \cdots z$, say m and n, are such that $m -\!\!<_o n$, then $\bar{n} -\!\!<_o \bar{m}$. If we further consider the order of elements in both lines, $o \cdots l \cdots z$, and $\bar{z} \cdots \bar{l} \cdots \bar{o}$, with reference to the origin o and its obverse $\bar{o}$, the two lines appear as a single line which passes from o to $\bar{o}$ through l, and from $\bar{o}$ back to o through $\bar{l}$. Let m and n be any two elements of $o \cdots l \cdots z$ such that $F(on/m)$. We have $m -\!\!<_o n$. Hence $\bar{n} -\!\!<_o \bar{m}$. But if we have $F(on/m)$, then also $O(on\bar{m})$ and so $F(\bar{o}m/n)$. Hence $n -\!\!<_m \bar{o}$. Thus any two elements, m and n, such that m is between o and n, are also such that n is between m and $\bar{o}$. From the transitivity of the illative relation, $m -\!\!<_o \bar{o}$. But if $m -\!\!<_o \bar{o}$, then from the above $m -\!\!<_o o$. Thus we have the continuous line, $o \cdots m \cdots n \cdots \bar{o} \cdots \bar{n} \cdots \bar{m} \cdots o$, or $\bar{o} \cdots \bar{n} \cdots \bar{m} \cdots o \cdots m \cdots n \cdots \bar{o}$, which has so far the character of the projective line with o as origin and $\bar{o}$ the point at infinity. And if m, n, r, occur in that order in one 'direction' from the origin, then $\bar{m}$, $\bar{n}$, $\bar{r}$, occur in that order in the 'opposite direction' from the origin.

Certain further characteristics of order in the system may be mentioned briefly. In general, lines such as those considered above may "intersect" any number of times. From the definition of obverses, $O(a\bar{a})$ and $O(c\bar{c})$ always hold. But by postulate I, if $O(a\bar{a})$, then $O(a\bar{a}\bar{p})$, and hence $F(a\bar{a}/p)$, for any element p. Similarly, if $O(c\bar{c})$, then $F(c\bar{c}/p)$. Thus collections consisting of the F-resultants of different pairs may have any number of elements in common. But in terms of such operations as were in question in the definitions of 'sums' and 'products,' sets of resultants may be determined such that they have one and only one element in common. Thus certain selected lines in the system intersect once and once only. There are any number of such sets.

In general, if any pair of elements in a set are obverses of one another, all the other elements of the set will be resultants of this pair, and their entire array will be "one-dimensional" so far as dimensionality may be attributed to such a collection. The problem of selecting sets suitable for any space form—any n-dimensional array—is the problem of selecting so that O-collections will be excluded. Such sets, containing no obverses, are the 'flat collections' of Kempe. As he pointed out, the excluded obverses will form an exactly similar set, so that 'spaces' come in pairs somewhat suggesting companion hemispheres. In terms of "flat collections," one-dimensional, two-dimensional, n-dimensional arrays, may be specified in any number of ways.

Once the order of the system Σ is generated in terms of O-relations and F-relations, the determination of such more specialized types of order is a problem of selection only. In the words of Professor Royce, "Wherever a linear series is in question, wherever an origin of coördinates is employed, wherever 'cause and effect,' 'ground and consequence,' orientation in space or direction of tendency in time are in question, the dyadic asymmetrical relations involved are essentially the same as the relation here symbolized by $p \prec_v q$. This expression, then, is due to certain of our best established practical instincts and to some of our best fixed intellectual habits. Yet it is not the only expression for the relations involved. It is in several respects inferior to the more direct expression in terms of *o*-relations. . . . When, in fact, we attempt to describe the relations of the system Σ merely in terms of the antecedent-consequent relation, we not only limit ourselves to an arbitrary choice of origin, but miss the power to survey at a glance relations of more than a dyadic, or triadic character."[1]

With this hasty and fragmentary survey of the system Σ, we may turn to considerations of method. It was suggested in the introduction that the procedure here exemplified differs in notable ways from the method of such studies as those of *Principia Mathematica*. In that work, we are presented at the outset with a simple, though general, order—the order of elementary propositions so related to one another that one is the negative of another, two may be such that at least one of them is true, and so on. In terms of these fundamental relations, more special types of order—various branches of mathematics—are built up by progressive complication. In some respects this is the necessary character of deductive procedures in general; in other respects it is not. In particular, this method differs from that employed by Mr. Kempe and Professor Royce in that *terms*, as well as relations, of later sections are themselves complexes of the relations at first assumed. The complication thus made necessary can hardly be appreciated by those who would regard a number, for instance, as a simple entity. To illustrate: In *Principia Mathematica*, the "cardinal number" of x is the class of referents of the relation 'similar to' where x is the relatum.[2] The 'class of referents' of any relation R is defined as α such that α is identical with x such that, for some y, x has the relation R to y. 'Relatum' is similarly defined. 'm is identical with n' means that, for any predicative function φ, φm implies φn. I do not pause upon 'predicative function.' α is 'similar to'

[1] Pages 381–2 of the paper.

[2] I shall, perhaps, be pardoned for translating the symbolism,—provided I do not make mistakes.

β means that, for some one-to-one relation R, α is identical with the class of referents of R and β is identical with the class of relata of R. A 'one-to-one' relation is a relation S such that the class of referents of S is contained in 1 and the class of relata of S is contained in 1.[1] '1' is defined as α such that, for some x, α is identical with *the x*. '*The x*' is my attempt to translate the untranslateable. The attempt to analyze 'is contained in' would require much more space than we can afford. But supposing the analysis complete, we discover that the 'cardinal number of x' is ———, where ——— is the definition first given, with all the terms in it replaced by *their* definition, the terms in these replaced by *their* definition, and so on. All this complexity is internal to the *terms* of arithmetic. *And only when this process is complete* can any properties or relations of 'the cardinal number of x' be demonstrated. An advantage of this method is that the step from one order to another 'based upon it' is always such as to make clear the connection between the two. It preserves automatically the hierarchic arrangement of various departments of exact thinking. The process of developing this hierarchy is tedious and taxes our analytic powers, but there is always the prospect of assured success if we can perform the initial analysis involved in the definitions. But the disadvantages of this complexity can hardly be overemphasized. It is forbidding to those whose interests are simply 'mathematical' or 'scientific' in the ordinary sense. Such a work as *Principia Mathematica* runs great risk of being much referred to, little read, and less understood.

In contrast with such complexity, we have, by the method of Mr. Kempe and Professor Royce, an order completely generated at the start, and such that the various special orders contained in it may be arrived at *simply by selection*. Little or no complication within the terms is required. Involved as the structure of the system Σ may seem, it is, by comparison, a marvel of simplicity and compact neatness. With this method,

[1] More accurately, "every member of the class of referents of S is contained in 1, and every member of the class of relata of S is contained in 1," because all relations are, in *Principia Mathematica*, taken in the abstract.

there seems to be no assurance in advance that any hierarchic relations of different orders will be disclosed, but we shall certainly discover, and without difficulty, whatever analogies exist between various orders. Again, this method relies much more upon devices which may be not at all obvious. It may not tax severely the analytic powers, but it is certain to tax the ingenuity.

In another important respect, advantage seems to lie with this method. One would hardly care to invent a new geometry by the hierarchic procedure, or expect to discover one by its use. We have to know where we are going or we shall not get there by this road. By contrast, Professor Royce's is the method of the path-finder. The prospect of the novel is here much greater. The system Σ may—probably does—contain new continents of order whose existence we do not even suspect. And some chance transformation may put us, suddenly and unexpectedly, in possession of such previously unexplored fields.

Which of the two methods will prove, in the end, more powerful, no one can say at present. The whole subject is too new and undeveloped. Certainly it is to be desired that the direct and exploratory method be increasingly made use of, and that the advantages of studying very general types of order, such as the system Σ, be better understood.

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